NORMAL PROJECTIVE SURFACES AND DYNAMICS OF AUTOMORPHISM GROUPS OF PROJECTIVE VARIETIES

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Summary

We present results for two topics.

In Chapters 1 and 2, we studied normal projective surfaces with only quotient singularities over the complex number field. Log del Pezzo surface plays the role as the "opposite" of surface of general type. The complete classification of log del Pezzo surfaces of Cartier index 3 and rank 2 is given in Theorem 1. Log Enriques surface is a generalization of K3 and Enriques surface. In Theorem 2, we classified all the rational log Enriques surfaces of rank 18 by giving concrete models for the realizable types of these surfaces.

In Chapter 3, we studied the relation between the geometry of a variety and its automorphism group. In particular, we prove some slightly finer Tits alternative theorems for automorphism groups of compact Kähler manifolds (Theorems 3.1, 3.2, 3.3), give sufficient conditions for the existence of equivariant fibrations of surfaces for the dimension reduction purpose (Theorem 3.4), determine the uniqueness of automorphisms on surface (Theorem 3.5), and confirm, to some extent, the belief that a compact Kähler manifold has lots of symmetries only when it is a torus or its quotient (Theorem 3.6).

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Chapter 1

Logarithmic del Pezzo Surfaces of Rank 2 and Cartier Index 3

1.1 Introduction

Del Pezzo surface (Definition 1.1) is the Fano variety of dimension two, which is one of the important topics in the classification theory of algebraic surfaces. It plays the role as the "opposite" of surface of general type.

The logarithmic (abbr. log) del Pezzo surface (Definition 1.2) is the del Pezzo surface with only logarithmic terminal singularities.

The open log del Pezzo surfaces of rank one are discussed by Miyanishi and Tsunoda in [24], [25], [26]; and the (complete) log del Pezzo surfaces of rank one are studied by Kojima [18], [19], Zhang [39], [40]. Alexeev and Nikulin give the classification of the log del Pezzo surfaces of index ≤ 2 in [1], and Nakayama gives a geometrical classification without using the theory of K3 lattices in [16].

Definition 1.1. A normal projective surface X is called a *del Pezzo surface* if its anti-canonical divisor $-K_X$ is ample.

Definition 1.2 ([40, Definition 1])**.** Let $\bar{X}$ be a normal projective surface with only quotient singularities. Then $\bar{X}$ is called a *logarithmic* (abbr. *log*) *del Pezzo surface* if its anti-canonical divisor $-K_{\bar{X}}$ is an ample $\mathbb{Q}$-Cartier divisor.

The smallest positive integer I such that $IK_{\bar{X}}$ is a Cartier divisor is called the *Cartier index* of $\bar{X}$, and the Picard number $\rho(\bar{X})$ is called the *rank* of $\bar{X}$.

For notations and terminologies, we refer to Section 1.2. In this chapter, we will give the complete classification of the log del Pezzo surfaces of rank 2 and Cartier index 3 with a unique singularity.

Theorem 1. *Let $\bar{X}$ be a log del Pezzo surface with a unique singularity x_0, and (X, D) the minimal resolution. Suppose that $\bar{X}$ has rank 2 and Cartier index 3. Then the following assertions hold:*

1) *There is a contraction $\pi : \bar{X} \to \bar{Y}$ of an irreducible curve $\bar{C}$ on $\bar{X}$ to a log del Pezzo surface of rank 1. The proper transform C of $\bar{C}$ on X is a (-1)-curve.*

2) *The weighted dual graph of $C+D$ is of one of the 29 configurations in Figure 1.6. Moreover, they are all realizable.*

1.2 Preliminaries

We work on an algebraically closed field of characteristic zero.

Definition 1.3. [16, Definition 0.2.10] Let $\bar{X}$ be a normal variety. Then $\bar{X}$ is said to have *log terminal singularities* if

1) the canonical divisor $K_{\bar{X}}$ is a $\mathbb{Q}$-Cartier divisor, i.e., $mK_{\bar{X}}$ is a Cartier divisor for some $m \in \mathbb{Z}^+$, and

2) there exists a resolution of singularities $f : X \to \bar{X}$ with irreducible exceptional divisors $\{D_j\}_{j=1}^n$ such that $D := \sum_{j=1}^n D_j$ is a simple normal crossing divisor, and that
$$K_X = f^*(K_{\bar{X}}) + \sum_{j=1}^n \alpha_j D_j$$
for some $\alpha_j \in \mathbb{Q}$ with $\alpha_j > -1$.

Lemma 1.4 (cf. [15, Theorem 9.6], [24, §4.1])**.** *Suppose $\bar{X}$ is a normal surface. Then $\bar{X}$ has only log terminal singularities if and only if $\bar{X}$ has only quotient singularities. Moreover, if this is the case, let $X \to \bar{X}$ be the minimal resolution, then each irreducible exceptional curve is a nonsingular rational curve.*

It follows from Definition 1.3 and Lemma 1.4 that, the log del Pezzo surface as in Definition 1.2 is equivalent to "the del Pezzo surface with only log terminal singularities".

Remark 1.5. Let $\bar{X}$ be a log del Pezzo surface. Since $\dim \bar{X} = 2$, in Definition 1.3 we can take $f : X \to \bar{X}$ to be the minimal solution. Then $\alpha_j \leq 0$ for all j. It follows that

$D^{\#} := -\sum_{j=1}^{n} \alpha_j D_j$ is an effective $\mathbb{Q}$-Cartier divisor, and $f^*(K_{\bar{X}}) = K_X + D^{\#}$. If $\alpha_k = 0$ for some k, then $\alpha_j = 0$ for all D_j in the connected component of D containing D_k ([22, Proposition 4-6-2]). If $D^{\#} = 0$, then $f^*(K_{\bar{X}}) = K_X$ and $\bar{X}$ is a Gorenstein log del Pezzo surface, which is completely classified in [36]. The case when $\bar{X}$ has index 2 is classified in [1] and [27].

Lemma 1.6. *Let $\bar{X}$ be a log del Pezzo surface. With the notations in Remark 1.5, we have the following assertions:*

1) $-(K_X + D^{\#}) \cdot C \geq 0$ *for every irreducible curve C on X, and the equality holds if and only if $C \subseteq \mathrm{Supp}(D)$.*

2) *If $C \nsubseteq \mathrm{Supp}(D)$ is an irreducible curve on X with negative self-intersection number, then C is a (-1)-curve.*

3) $\rho(X) = n + \rho(\bar{X})$, *where n is the number of irreducible curves of the exceptional divisor of $f : X \to \bar{X}$.*

Proof. 1) Note that f is birational. Since $-K_{\bar{X}}$ is ample,

$$-(K_X + D^{\#}) \cdot C = -f^*(K_{\bar{X}}) \cdot C = -K_{\bar{X}} \cdot f_*(C) \geq 0.$$

The equality holds if and only if $f_*(C)$ is a point, i.e., $C \subseteq \mathrm{Supp}(D)$.

2) Suppose $C \nsubseteq \mathrm{Supp}(D)$. Then by (1) and the adjunction formula,

$$0 < -(K_X + D^{\#}) \cdot C \leq -K_X \cdot C = 2 + C^2 - 2p_a(C) \leq 2 + C^2 \leq 1.$$

It follows that $C^2 = -1$ and $p_a(C) = 0$. So C is a (-1)-curve.

3) $\mathrm{NS}_{\mathbb{Q}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by $f^*(\mathrm{NS}_{\mathbb{Q}}(\bar{X}))$ and $\{D_j\}_{j=1}^n$. $\square$

In [18], (X, D) is assumed to be *almost minimal*, and we will show in the following that the minimal resolution of every log del Pezzo surface of rank 1 is almost minimal. Hence, we can use the classification for discussion in Sections 1.3–1.5.

Definition 1.7. [24, §3.11] Let $\bar{X}$ be a surface and $(X, D) \to \bar{X}$ the minimal resolution. With the notations in Remark 1.5, let $\mathrm{Bk}(D) = D - D^{\#}$. Then (X, D) is called *almost minimal* if for every irreducible curve C on X either

1) $(K_X + D^{\#}) \cdot C \geq 0$; or

2) the intersection matrix of $C + \mathrm{Bk}(D)$ is not negative definite.

Lemma 1.8. *Let $\bar{X}$ be a log del Pezzo surface of rank 1. Then its minimal resolution (X, D) is almost minimal.*

Proof. Suppose there exists an irreducible curve E on X such that $E \cdot (K_X + D^{\#}) < 0$ and the intersection matrix of $E + \mathrm{Bk}(D)$, i.e., of $E + D$, is negative definite.

Let $\bar{E} = f_*(E)$. Since $0 > E \cdot f^*(K_{\bar{X}}) = \bar{E} \cdot K_{\bar{X}}$, $\bar{E}$ is a curve on $\bar{X}$. Recall that $\rho(\bar{X}) = 1$. We can write $\bar{E} \equiv rK_{\bar{X}}$ for some $r \in \mathbb{Q}$. Then $(\bar{E})^2 = r^2(K_{\bar{X}})^2 \geq 0$.

On the other hand,

$$f^*(\bar{E}) = E + \sum_{j=1}^{n} \beta_j D_j$$

for some $\beta_j \in \mathbb{Q}$. Let $H = \sum_{j=1}^{n} \beta_j D_j$. Then

$$(\bar{E})^2 = (f^*(\bar{E}))^2 = (E + H)^2 < 0$$

because the intersection matrix of $E+D$ is negative definite. This leads to a contradiction. $\square$

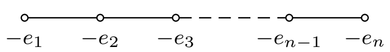
1.3 The Types of Weighted Dual Graphs of D

In this section, we assume that $\bar{X}$ is a log del Pezzo surface of Cartier index 3 with a unique singularity x_0, and use the notations in Section 1.2.

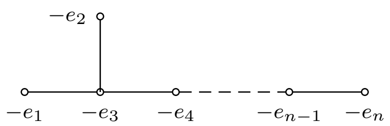
Recall that the exceptional divisor $D = \sum_{j=1}^{n} D_j$ is a connected simple normal crossing divisor. It can be drawn as a graph: each curve D_j is represented by a node, and intersecting curves D_i and D_j are joined by an edge; the node corresponding to D_i is marked with $-e_j := (D_j)^2$. This is known as the *weighted dual graph* of D.

Note that the intersection matrix $\{D_i \cdot D_j\}$ is negative definite. The dual graph of D is of one of the following A-D-E Dynkin's type (cf. [4, Lemma 2.12]):

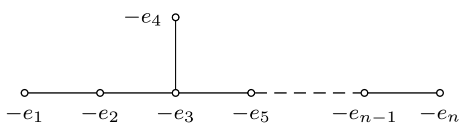
A_n:



D_n:



E_n:



$(n = 6, 7, 8)$

We are going to determine all the possible types of weighted dual graphs of D.

Let $a_j = -\alpha_j$. Then $f^*(K_{\bar{X}}) = K_X + \sum_{j=1}^{n} a_j D_j$ for some $0 < a_j < 1$. It is given

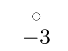
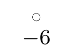
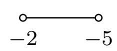
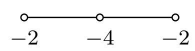
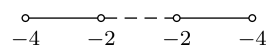
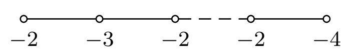
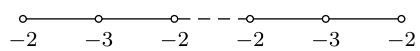
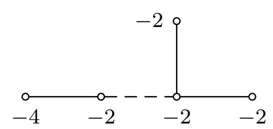
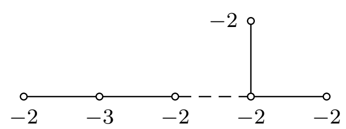
No.	Weighted Dual graph of D	Size
I	 -3	$n = 1$
II	 -6	$n = 1$
III	 -2 -5	$n = 2$
IV	 -2 -4 -2	$n = 3$
V	 -4 -2 -2 -4	$2 \leq n \leq 10$
VI	 -2 -3 -2 -2 -4	$3 \leq n \leq 9$
VII	 -2 -3 -2 -2 -3 -2	$4 \leq n \leq 9$
VIII	 -2 -4 -2 -2 -2	$4 \leq n \leq 9$
IX	 -2 -2 -3 -2 -2 -2	$4 \leq n \leq 8$

Figure 1.1: Weighted Dual graph of D

that $3K_{\bar{X}}$ is a Cartier divisor, so is $\sum_{j=1}^{n} 3a_j D_j$. Therefore, $a_j \in \{1/3, 2/3\}$ for all j.

Note that for each $i = 1, \ldots, n$,

$$0 = f^*(K_{\bar{X}}) \cdot D_i = \left(K_X + \sum_{j=1}^{n} a_j D_j \right) \cdot D_i = -2 - (D_i)^2 + \sum_{j=1}^{n} a_j (D_i \cdot D_j).$$

That is,

$$\sum_{j=1}^{n} a_j (D_i \cdot D_j) = 2 + (D_i)^2, \quad i = 1, \ldots, n.$$

Using these results, we can show that

Proposition 1.9. *Let $\bar{X}$ be a log del Pezzo surface of Cartier index 3 with a unique singularity, and (X, D) its minimal resolution. Then*

1) *the weighted dual graph of D is of one of the nine cases listed in the second column of Figure 1.1, and*

2) *the possible sizes of D are given in the third column of Figure 1.1.*

We will leave the proof of (2) in Section 1.4.

Proof of Proposition 1.9 (1). Consider the two cases:

Type A Suppose that D is a linear chain $D_1 - D_2 - \cdots - D_n$.

If $n = 1$, then $a_1(D_1)^2 = 2 + (D_1)^2$. When $a_1 = 1/3$, $(D_1)^2 = -3$, and D is given by I of Figure 1.1; when $a_1 = 2/3$, $(D_1)^2 = -6$, and D is given by II.

Suppose $n \geq 2$. Then for all $i = 2, \ldots, n$, $a_{i-1} + a_i(D_i)^2 + a_{i+1} = 2 + (D_i)^2$. This implies $2 - a_{i-1} - a_{i+1} = (D_i)^2(a_i - 1) \geq -2(a_i - 1)$, i.e.,

$$a_i \geq \frac{1}{2}(a_{i-1} + a_{i+1}).$$

Moreover, the equality holds if and only if $(D_i)^2 = -2$.

If $a_i = 1/3$ for some $i = 2, \ldots, n-1$, then $a_{i-1} + a_{i+1} \leq 2/3$ and thus $a_{i-1} = a_{i+1} = 1/3$; consequently $a_j = 1/3$ for all $j = 1, \ldots, n$. In particular, $1/3\,(D_1)^2 + 1/3 = 2 + (D_1)^2$. However, this would imply that $(D_1)^2 = -5/2 \notin \mathbb{Z}$, a contradiction.

So $a_i = 2/3$ for some $i = 2, \ldots, n-1$. If $i \leq n-2$, then $a_{i+1} \geq \frac{1}{2}(a_i + a_{i+2}) \geq \frac{1}{2}(\frac{2}{3} + \frac{1}{3}) = 1/2$, and then $a_{i+1} = 2/3$. It follows by induction that $a_j = 2/3$ for all $j = i, \ldots, n-1$; and similarly $a_j = 2/3$ for all $j = 2, \ldots, i$. We consider three cases:

(i) $a_j = 2/3$ for all $j = 1, \ldots, n$. Then $(D_1)^2 = (D_n)^2 = -4$ and $(D_j)^2 = -2$ for $j = 2, \ldots, n-1$. This is given by V of Figure 1.1.

(ii) $a_1 = 1/3$ and $a_j = 2/3$ for all $j = 2, \ldots, n$. For this case, if $n = 2$, then $(D_1)^2 = -2$ and $(D_2)^2 = -5$, which is given by III; if $n \geq 3$, then $(D_2)^2 = -3$, $(D_n)^2 = -4$ and $(D_j)^2 = -2$ for all other j, which is given by VI of Figure 1.1.

(iii) $a_1 = a_n = 1/3$ and $a_j = 2/3$ for all $j = 2, \ldots, n-1$. It is impossible if $n = 2$. If $n = 3$, then $(D_1)^2 = (D_3)^2 = -2$ and $(D_2)^2 = -4$, which is given by IV; if $n \geq 4$, then $(D_2)^2 = (D_{n-1})^2 = -3$ and $(D_j)^2 = -2$ for all other j, which is given by VII.

Type D and E Suppose that D is a fork. Let D_3 be the center of the fork. It intersects with three components, say D_1, D_2 and D_4. Then $a_1 + a_3 + a_4 + a_2(D_2)^2 = 2 + (D_2)^2$.

If $(D_3)^2 \leq -3$, then $1 \geq 2 - a_1 - a_2 - a_4 = (D_3)^2(a_3 - 1) \geq (-3)(1/3) = 1$. We have $a_1 = a_2 = a_4 = 1/3$, $a_3 = 2/3$ and $(D_3)^2 = -3$. If D_4 intersects with, say, D_5, then $2/3 + a_5 + 1/3\,(D_4)^2 = 2 + (D_4)^2$ implies $(D_4)^2 = (3/2)a_5 - 2 \geq -3/2$, a contradiction. So D_4 is the end of a twig, and the same is true for D_1 and D_2. Therefore, for this case $n = 4$ and $(D_1)^2 = (D_2)^2 = (D_4)^2 = -2$. The weighted dual graph is by IX ($n = 4$).

Suppose $(D_3)^2 = -2$. Then $a_1 + a_2 + a_4 = 2a_3$. It follows that $a_3 = 2/3$ and $a_1 + a_2 + a_4 = 4/3$. After the relabeling if necessary, we have $a_1 = a_2 = 1/3$ and $a_4 = 2/3$. Using the same argument as above, D_1 and D_2 are twigs of D consisting of a single (-2)-curve.

We are left to determine the last twig of D: $\begin{matrix} D_1 \\ D_2 \end{matrix}\!\!> D_3 - D_4 - \cdots - D_n$. Using the same argument as in the case of linear chain, it follows by induction that $a_j = 2/3$ for all $j = 4, \ldots, n-1$. There are two cases:

(i) $a_1 = a_2 = 1/3$ and $a_j = 2/3$ for all $j = 3, 4, \ldots, n$. Then $(D_n)^2 = -4$ and $(D_j)^2 = -2$ for all $j = 1, \ldots, n-1$. This is given by VIII of Figure 1.1.

(ii) $a_1 = a_2 = a_n = 1/3$ and $a_j = 2/3$ for all $j = 3, 4, \ldots, n-1$. Then $n \geq 5$, $(D_{n-1})^2 = -3$ and $(D_j)^2 = -2$ for all $j \neq n-1$. This is given by IX $(n \geq 5)$. $\square$

1.4 Contraction

From now till the end of this chapter, we assume that $\bar{X}$ is a log del Pezzo surface of rank 2 and Cartier index 3 with a unique singularity x_0.

Since $K_{\bar{X}}$ is not numerically effective, by cone theorem, there is a $K_{\bar{X}}$-negative extremal ray $R \subseteq \overline{NE}(\bar{X})$. Let $\pi : \bar{X} \to \bar{Y}$ be the contraction of R. Then $\bar{Y}$ is a normal projective variety of $\dim \bar{Y} \leq 2$ and π has connected fibers. We will consider the three possibilities according to the dimension of $\bar{Y}$.

Case 1: $\dim \bar{Y} = 0$. It follows that $N_1(\bar{X})$ is generated by some $[\bar{C}] \in R$, and thus $\rho(\bar{X}) = 1$. But we assumed that $\rho(\bar{X}) = 2$, a contradiction.

Case 2: $\dim \bar{Y} = 1$. Then $\dim(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ implies that $\dim(\bar{Y}, \mathcal{O}_{\bar{Y}}) = 0$, i.e., $\bar{Y} \cong \mathbb{P}^1$. We claim that

Lemma 1.10. *With the notations above, every fiber of the contraction* $\pi : \bar{X} \to \bar{Y}$ *is irreducible.*

Proof. Since $\bar{Y}$ is nonsingular, the contraction $\pi : \bar{X} \to \bar{Y}$ is flat, and thus every fibre has pure dimension 1.

For any point $y \in \bar{Y}$, let $\bar{F} = \pi^{-1}(y)$. Suppose $\bar{F}$ is reducible. Since $\bar{F}$ is connected, we may choose irreducible components $\bar{F}_1$ and $\bar{F}_2$ of $\bar{F}$ such that $\bar{F}_1 \cdot \bar{F}_2 \geq 1$. On the other hand, $\bar{F}_1 \equiv a\bar{F}_2 \in R$ for some $a > 0$. Then by Zariski's lemma (cf. [4, Lemma 8.2]), $\bar{F}_1 \cdot \bar{F}_2 = a(\bar{F}_2)^2 < 0$, a contradiction. □

Let $y_0 = \pi(x_0)$ and $\bar{C} = \pi^{-1}(y_0)$. Then $x_0 \in \bar{C}$, and by Zariski's lemma, $(\bar{C})^2 = 0$.

Take $f : (X, D) \to \bar{X}$ to be be the minimal resolution, and C the proper transform of $\bar{C}$ with respect to f. Then $C + D = (\pi \circ f)^{-1}(y_0)$. By Zariski's lemma again, $C^2 < 0$, and thus C is a (-1)-curve by Lemma 1.6.

Let $y \in \bar{Y} \backslash \{y_0\}$, $\bar{F} := \pi^{-1}(y)$ and F the proper transform of $\bar{F}$ with respect to f. Then $F = (\pi \circ f)^{-1}(y)$. So $F^2 = 0$ and $F \cdot D^{\#} = 0$. We have

$$0 > \bar{F} \cdot K_{\bar{X}} = F \cdot (K_X + D^{\#}) = F \cdot K_X.$$

Then by adjunction formula, $2p_a(F) - 2 = F \cdot (F + K_X) = F \cdot K_X < 0$, and thus $p_a(F) = 0$. By Lemma 1.10, F is irreducible; so $F \cong \mathbb{P}^1$.

Let F_0 be the singular fiber of the the $\mathbb{P}^1$-fibration $\pi \circ f : X \to \bar{Y}$ over y_0. Then $\mathrm{Supp}(F_0) = C + D$. After contracting C and consecutively (-1)-curves in $C + D$, $C + D$ becomes $\mathbb{P}^1$. In particular, note that D is connected and $C + D$ is a connected simple normal crossing divisor, we have $C \cdot D = 1$. Moreover,

$$2 + n = \rho(X) = 10 - (K_X)^2. \tag{1.1}$$

Case 3: $\dim \bar{Y} = 2$. Then $\pi : \bar{X} \to \bar{Y}$ is birational and the exceptional curve is irreducible [20, Proposition 2.5], denoted by $\bar{C}$. Let C be the proper transform of $\bar{C}$ with respect to the minimal resolution $f : (X, D) \to \bar{X}$.

Note that $\pi \circ f : X \to \bar{Y}$ contracts C into a point. By negative definiteness theorem, $C^2 < 0$. So by Lemma 1.6, C is a (-1)-curve.

By [16, Proposition 5-1-6], $\bar{Y}$ is $\mathbb{Q}$-factorial, and it is either smooth or it has a unique log terminal singularity $y_0 = \pi(x_0)$. By taking $H = -K_{\bar{X}}$ in Lemma 1.11 below, $-K_{\bar{Y}}$ is ample. Therefore, $\bar{Y}$ is either a smooth del Pezzo surface or a log del Pezzo surface with a unique singularity y_0. Recall that $\rho(\bar{Y}) = 1$. If $\bar{Y}$ is smooth, then $\bar{Y} \cong \mathbb{P}^2$, the projective plane.

Lemma 1.11. *With the notations as above, for any ample divisor H on $\bar{X}$, $\pi_*(H)$ is ample.*

Proof. Let $\bar{H} = \pi_*(H)$. Then by projection formula $H = \pi^*(\bar{H}) + a\bar{C}$ for some $a \in \mathbb{Q}$.

Suppose $x_0 \in \bar{C}$. Since $f^{-1}(\bar{C}) = \mathrm{Supp}(C + D)$ and that the intersection matrix of $C + D$ is negative definite, $(\bar{C})^2 = (f^*(\bar{C}))^2 < 0$. If $x_0 \notin \bar{C}$, then $(\bar{C})^2 = C^2 = -1$. For either case,

$$0 < H^2 = (\pi^*(\bar{H}) + a\bar{C})^2 = (\pi^*(\bar{H}))^2 + a^2(\bar{C})^2 \leq (\pi^*(\bar{H}))^2 = (\bar{H})^2.$$

Let $\bar{E}$ be an irreducible curve on $\bar{Y}$ and $\bar{E}'$ the proper transform of $\bar{E}$ with respect to π. Then $\pi^*(\bar{E}) = \bar{E}' + b\bar{C}$ for some $b \in \mathbb{Q}$. We can compute that

$$0 = \bar{C} \cdot \pi^*(\bar{E}) = \bar{C} \cdot \bar{E}' + b(\bar{C})^2 \geq b(\bar{C})^2.$$

So $b \geq 0$. Then

$$\bar{H} \cdot \bar{E} = H \cdot \pi^*(\bar{E}) = H \cdot (\bar{E}' + b\bar{C}) = H \cdot \bar{E}' + b(H \cdot \bar{C}) \geq H \cdot \bar{E}' > 0.$$

By Nakai-Moishezon criterion, $\bar{H}$ is an ample divisor on $\bar{Y}$. ◻

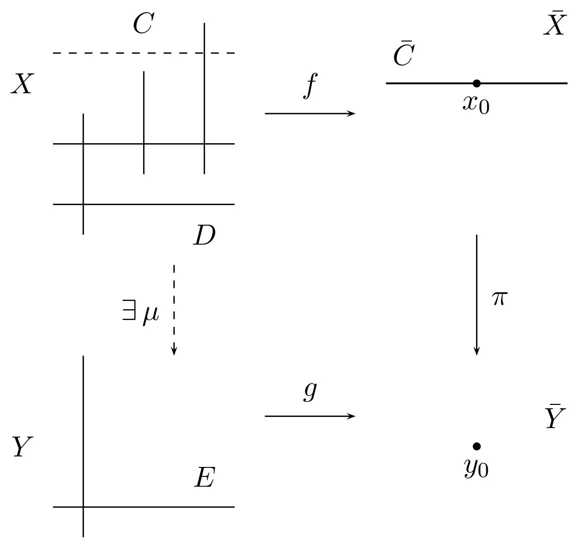


Figure 1.2: Divisorial Contraction

Let $g : Y \to \bar{Y}$ be the minimal resolution. Then $\pi \circ f$ factors through Y; that is, there is a proper birational morphism $\mu : X \to Y$ such that $g \circ \mu = \pi \circ f$ as illustrated in Figure 1.2. We see that $\mu : X \to Y$ is the composite of blow-downs of (-1)-curves. More precisely, it is the contraction of C and consecutive (-1)-curves in $C + D$.

Let $y_0 = f(x_0)$. If $\bar{Y} \cong \mathbb{P}^2$, then $Y = \bar{Y}$ and $\mu(C + D) = y_0$. Suppose $\bar{Y}$ is a log del Pezzo surface of rank 1 with a unique singularity y_0. Then Y can be further contracted along (-1)-curves into the Hirzebruch surface $\mathbb{F}_r$ for some $r \geq 0$ [18, Theorem 2.1, 3.1, 4.1]. For either case,

$$2 + n = \rho(X) = 10 - (K_X)^2. \tag{1.2}$$

We can now determine the size of the weighted dual graphs of D in Figure 1.1.

Proof of Proposition 1.9 (b). Recall that $-K_{\bar{X}}$ is ample. In particular,

$$\begin{aligned} 0 < (K_{\bar{X}})^2 &= \pi^*(K_{\bar{X}}) \cdot \pi^*(K_{\bar{X}}) = K_X \cdot \pi^*(K_{\bar{X}}) \\ &= K_X \cdot \left(K_X + \sum_{j=1}^{n} a_j D_j \right) \\ &= (K_X)^2 + \sum_{j=1}^{n} a_j(-2 - (D_j)^2). \end{aligned}$$

For both the fiber contraction (1.1) and the divisorial contraction (1.2),

$$2 + n = \rho(X) = 10 - (K_X)^2 < 10 + \sum_{j=1}^{n} a_j(-2 - (D_j)^2).$$

That is, $n < 8 + \sum_{j=1}^{n} a_j(-2 - (D_j)^2)$. Recall that $D^{\#} = \sum_{j=1}^{n} a_j D_j$ is evaluated explicitly in the proof of part (a), we can easily compute the possible size n of D:

V. $n < 8 + 2/3 \cdot 2 + 2/3 \cdot 2 \Leftrightarrow n \leq 10$;

VI. $n < 8 + 2/3 \cdot 1 + 2/3 \cdot 2 \Leftrightarrow n \leq 9$;

VII. $n < 8 + 2/3 \cdot 1 + 2/3 \cdot 1 \Leftrightarrow n \leq 9$;

VIII. $n < 8 + 2/3 \cdot 2 \Leftrightarrow n \leq 9$;

IX. $n < 8 + 2/3 \cdot 1 \Leftrightarrow n \leq 8$.

This completes the proof of Proposition 1.9. $\square$

Proof of Theorem 1. 1) Suppose $\dim \bar{Y} = 1$. We have seen that $C + D$ can be smoothly contracted to $F \cong \mathbb{P}^1$ with $F^2 = 0$ along C and consecutive (-1)-curves in $C + D$. However, by verifying all the weighted dual graphs in Figure 1.1, none of them with any (-1)-curve can be contracted to such a curve, a contradiction.

Therefore, $\dim \bar{Y} = 2$ and $\bar{Y}$ is a log del Pezzo surface of rank 1. In particular, as proved in Section 1.4, C is a (-1)-curve.

2) *Case 1.* If $\bar{Y}$ is smooth, then $Y = \bar{Y} \cong \mathbb{P}^2$ and $C + D$ is contracted to the smooth point y_0 along C and consecutive (-1)-curves in $C + D$. In particular, by noting that D is a simple normal crossing divisor, we have $C \cdot D = 1$.

Case 2. Suppose $\bar{Y}$ is not smooth. Then $\bar{Y}$ is a log del Pezzo surface with a unique singularity y_0. Let E be the exceptional divisor of the minimal resolution $g : Y \to \bar{Y}$. The configuration of E is completely classified in [18, Theorem 2.1]. Recall that the possible weighted dual graphs of D have been listed in Figure 1.1.

(i) If $x_0 \notin \bar{C}$, then C is disjoint from D, and the weighted dual graphs of D is the same as that of E.

(ii) If $x_0 \in \bar{C}$, then $C + D$ is a connected simple normal crossing divisor since E is of A-D-E Dynkin's type. Note that D is connected. Then $C \cdot D = 1$ and $X \backslash (C \cup D) \cong Y \backslash E$. We only need to check how $C + D$ is contracted to E along C

and consecutive (-1)-curves in $C + D$.

By checking all the possible weighted dual graphs of D in Figure 1.1 and all the possible places of C, there are 3 configurations of $C + D$ (VI ($n = 5$) (b), VI ($n = 6$) (b), IX ($n = 5$) (b)) for the case when $\bar{Y}$ is smooth, and 26 configurations of $C + D$ for the case when $\bar{Y}$ is not smooth. They are given in Figure 1.6.

According to the discussions above, each of these 29 possible configurations of $C + D$ can be contracted to E (resp. a smooth point) along consecutive (-1)-curves in $C + D$. There exists a log del Pezzo surface $\bar{Y}$ of rank 1 with a unique singularity (resp. $\bar{Y} \cong \mathbb{P}^2$), such that E is the exceptional divisor of its minimal resolution $Y \to \bar{Y}$ (resp. $Y = \bar{Y}$). We can construct the surface X by blowing up points from the corresponding surface Y. Let $X \to \bar{X}$ be the contraction of D. Then $\bar{X}$ is a projective normal surface of rank 2 and Cartier index 3 with a unique quotient singularity. We claim that

Lemma 1.12. *For each of the configuration of $C + D$ in Figure 1.6, let $\bar{X}$ be the surface defined above, then $-K_{\bar{X}}$ is ample.*

It follows that $\bar{X}$ is a log del Pezzo surface of rank 2 and Cartier index 3 with a unique singularity x_0, and D is the exceptional divisor of its minimal resolution $X \to \bar{X}$. In other words, every configuration in Figure 1.6 is realizable. We have completed the proof of Theorem 1. □

1.5 Ampleness of $-K_{\bar{X}}$

In the proof of Theorem 1, for each weighted graph of $C + D$ in Figure 1.6, we constructed a normal projective surface $\bar{X}$ of rank 2 and Cartier index 3 with a unique quotient singularity, such that D is the exceptional divisor of its minimal resolution $X \to \bar{X}$. In order to prove that $\bar{X}$ is a log del Pezzo surface, it remains to show that $-K_{\bar{X}}$ is ample.

First of all, we shall evaluate $-K_{\bar{X}}$. We explore the notations used in the discussion of the divisorial contraction case in Section 1.4 (as illustrated in Figure 1.2). Recall that $\mu : X \to Y$ is the successive contraction of (-1)-curves in $C+D$. If $\bar{Y}$ is smooth, then $Y = \bar{Y} \cong \mathbb{P}^2$, and μ factors through $X \to \mathbb{F}_1 \to Y$. If $\bar{Y}$ has a unique singularity, then Y can be further contracted to the Hirzebruch surface $\mathbb{F}_r$ for some $r \geq 0$ along (-1)-curves [18, Theorem 3.1, 4.1].

We can verify the list of configurations in Figure 1.6 to conclude that

Lemma 1.13. *Let $\bar{X}$ be a log del Pezzo surface of rank 2 and Cartier index 3 with a unique singularity, and $(X, D) \to X$ the minimal resolution. Then there exists a $\mathbb{P}^1$-fibration $X \xrightarrow{\Phi} \mathbb{F}_r \to \mathbb{P}^1$ with at most two singular fibers, such that one of the component D_ℓ of D is a cross-section, C and the other components of D are contained in the singular fibers.*

Then $M_r := \Phi(D_\ell)$ is the minimal section of $\mathbb{F}_r$. If there are two singular fibers, let their images in $\mathbb{F}_r$ be F_1 and F_2. If there is only one singular fiber, let its image in $\mathbb{F}_r$ be F_1 and take F_2 to be the image of a general fiber. Take a section $N_r \sim M_r + rF_1$ which does not contain the image of any center of blowups. Then $-K_{\mathbb{F}_r} = M_r + N_r + F_1 + F_2$, which form a circle (Figure 1.3).

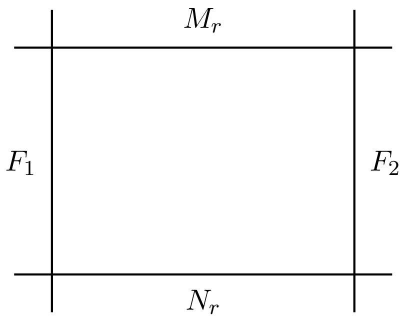


Figure 1.3: $-K_{\mathbb{F}_r}$

We can decompose $\Phi : X \to \mathbb{F}_r$ as the composite of blow-downs $X = X_0 \xrightarrow{\phi_1} X_1 \to \cdots \to X_{k-1} \xrightarrow{\phi_k} X_k = \mathbb{F}_r$. Denote the exceptional curve of ϕ_i by E_i, $i = 1, \ldots, k$. Then $K_{X_{i-1}} = \phi_i^*(K_{X_i}) + E_i$. Therefore, $-K_X$ can be evaluated explicitly.

Note that $-K_X$ is supported by $\Delta := \Phi^{-1}(M_r + N_r + F_1 + F_2)$. Let Δ_+ denote the sum of the irreducible curves which have positive coefficients appearing in $-K_X$. Note that Δ_+ forms a loop, and every irreducible curve in Δ_+ has coefficient 1 appearing in $-K_X$. In particular, the proper transforms of M_r, N_r, F_1 and F_2 on X belong to Δ_+.

Recall that in the proof of Proposition 1.9 (1), we computed the unique numbers $a_j \in \{1/3, 2/3\}$, $i = 1, \ldots, n$, such that

$$f^*(K_{\bar{X}}) = K_X + \sum_{j=1}^{n} a_j D_j.$$

We can thus evaluate $-f^*(K_{\bar{X}})$ explicitly.

The weighted dual graphs for some $-f^*(K_{\bar{X}})$ are illustrated in Figures 1.4 and 1.5. For each of the irreducible curve, the label with brackets indicates its coefficient, and that without brackets indicates its self-intersection number. The labels of coefficient 1 are omitted. A dotted line stands for a (-1)-curve, and a solid line stands for a (-2)-curve if its self-intersection number is not indicated.

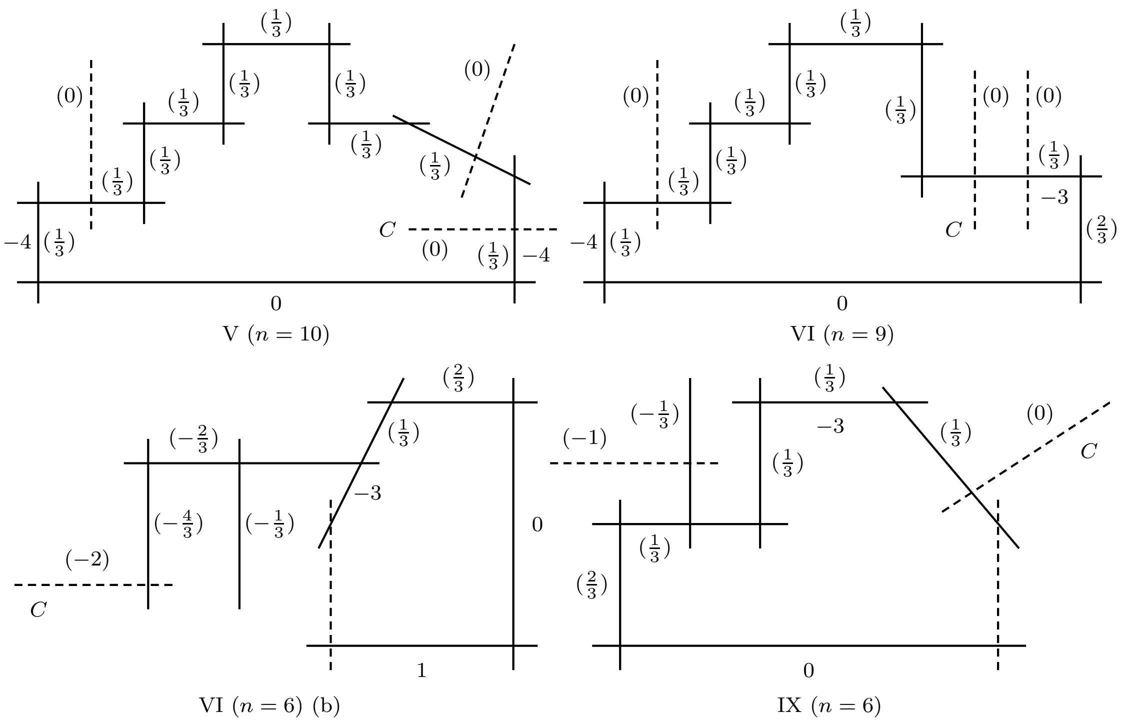
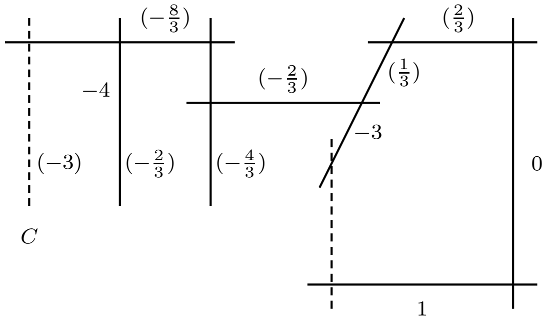


Figure 1.4: $-f^*(K_{\bar{X}}) \quad (c_1 + c_2 + r = 0)$

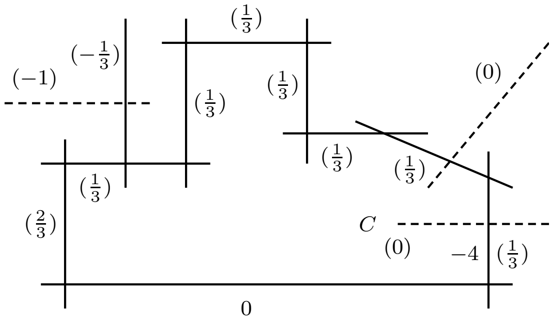
Proof of Lemma 1.12. From the proof of Proposition 1.9 (2),

$$(-K_{\bar{X}})^2 = (K_X)^2 + \sum_{j=1}^{n} a_j(-2 - (D_j)^2) = 8 - n + \sum_{j=1}^{n} a_j(-2 - (D_j)^2).$$

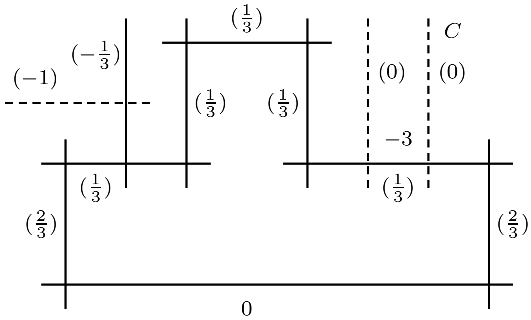
The size n of D in Figure 1.1 is chosen so that $n > 8 + \sum_{j=1}^{n} a_j(-2 - (D_j)^2)$. Then $(-K_{\bar{X}})^2 > 0$. So by Nakai-Moishezon criterion, $-K_{\bar{X}}$ is ample if and only $-K_{\bar{X}} \cdot \bar{G} > 0$ for every irreducible curve $\bar{G}$ on $\bar{X}$.



VI ($n = 6$) (b)



VIII ($n = 9$)



IX ($n = 8$)

Figure 1.5: $-f^*(K_{\bar{X}}) \quad (c_1 + c_2 + r < 0)$

Let $\bar{G}$ be an irreducible curve on $\bar{X}$, and G the proper transform of $\bar{G}$ on X. Then

$$-K_{\bar{X}} \cdot \bar{G} = -f^*(K_{\bar{X}}) \cdot f^*(\bar{G}) = -f^*(K_{\bar{X}}) \cdot G.$$

We will show that this number is positive by considering the following two possibilities.

G is contained in a fiber.

Case 1. Suppose G is a general fiber. Then $\bar{G}$ does not contain the image of any center of blowup. So G intersects with the proper transforms of M_r and N_r on X. It follows that $-f^*(K_{\bar{X}}) \cdot G \geq 1 + 1/3 > 0$.

Case 2. Suppose G is contained in a singular fiber. Then $G^2 < 0$. Note that

$G \nsubseteq \mathrm{Supp}(D)$. By Lemma 1.6, G is a (-1)-curve. Its coefficient in $-f^*(K_{\bar{X}})$ is the same as that in $-K_X$.

(i) If $G \subseteq \mathrm{Supp}(\Delta_+)$, then G intersects with exactly two irreducible components of Δ, which are contained in Δ_+. Moreover, exactly one of them is an irreducible component of D. We have $-f^*(K_{\bar{X}}) \cdot G \geq (-1) + 1/3 + 1 > 0$.

(ii) If $G \nsubseteq \mathrm{Supp}(\Delta_+)$, let c be the coefficient of G in $-K_X$, then G intersects with exactly one irreducible component of D, whose coefficient in $-K_X$ is $c+1$. Note that G is disjoint from any other irreducible component of Δ. So $-f^*(K_{\bar{X}}) \cdot G \geq (-1)c + (c + 1 - 2/3) > 0$.

G is not contained in a fiber.

Note that $G_0 := \Phi(G)$ is a curve in $\mathbb{F}_r$. Write $G_0 \sim aM_r + bF_1$, where $a > 0$ and $b \geq ar$. We have $G_0 \cdot F_1 = G_0 \cdot F_2 = a$, $G_0 \cdot M_r = b - ar \geq 0$ and $G_0 \cdot N_r = b$.

Let c_i be the smallest coefficient among all the irreducible components of $\Phi^{-1}(F_i)$ appearing in $-f^*(K_{\bar{X}})$, $i = 1, 2$. Then

$$-f^*(K_{\bar{X}}) \cdot G \geq ac_1 + ac_2 + 0 + b \geq a(c_1 + c_2 + r). \tag{1.3}$$

By considering the sign of $c_1 + c_2 + r$, we have the following three cases:

Case 1. $c_1 + c_2 + r > 0$. This is true for 22 configurations in Figure 1.6. For this case, it follows immediately from (1.3) that $-f^*(K_{\bar{X}}) \cdot G > 0$.

Case 2. $c_1 + c_2 + r = 0$. There are 4 configurations as given in Figure 1.4.

For this case, we may assume that $b = ar$; otherwise $b > ar$ and (1.3) implies that $-f^*(K_{\bar{X}}) \cdot G \geq a(c_1 + c_2 + r) + (b - ar) > 0$. Then $G_0 \sim aN_r$, and thus G_0 is disjoint from the minimal section M_r. Therefore, there must exist irreducible curves $L_i \subseteq \Phi^{-1}(F_i)$ with coefficient c_i appearing in $-f^*(K_{\bar{X}})$ such that $\Phi(L_i)$ is not a point in M_r ($i = 1, 2$). However, it is easy to see from Figure 1.4 that F_1 does not exist for any of these 4 configurations.

Case 3. $c_1 + c_2 + r < 0$. There are 3 configurations as given in Figure 1.5.

For each of them, denote $\{P_i\} := M_r \cap F_i$ ($i = 1, 2$), and let C', C'' be the irreducible curves in $\Phi^{-1}(F_1)$ with coefficients $\leq -(c_2 + r)$ in $-f^*(K_{\bar{X}})$. Suppose that $-f^*(K_{\bar{X}}) \cdot G \leq 0$. Then $s := (C' + C'') \cdot G > 0$.

(i) VI ($n = 6$) (b). By computing the multiplicities of the center of blowups, we have $(F_1 \cdot G_0)_{P_1} \geq 4s$ and $(M_1 \cdot G_0)_{P_1} \geq 4s$. In particular, $G_0 \sim aM_1 + bF_1$ with $a \geq 4s$ and $b \geq 8s$. Then it would follow that $-f^*(K_{\bar{X}}) \cdot G \geq (-3)s + 4s + 8s > 0$, a contradiction.

(ii) and (iii). VIII ($n = 9$) and IX ($n = 8$). For these cases, $(M_0 \cdot G_0)_{P_1} \geq s$ and $(F_1 \cdot G_0)_{P_1} \geq 2s$. If $P_2 \in F_2 \cap G_0$, then $G_0 \cdot N_0 \geq (G_0 \cdot M_0)_{P_1} + (G_0 \cdot M_0)_{P_2} \geq s + 1$. We would have $-f^*(K_{\bar{X}}) \cdot G \geq (-1)s + (s + 1) > 0$. Suppose $P_2 \notin F_2 \cap G_0$.

IX ($n = 8$): Let F_2' be the proper transform of F_2 on X. Then $G \cdot F_2' = G_0 \cdot F_2 \geq 2s$. But then $-f^*(K_{\bar{X}}) \cdot G \geq (-1)s + (2/3)2s + s > 0$, a contradiction.

VII ($n = 9$): Note that $-f^*(K_{\bar{X}}) \cdot G \geq (-1)s + s = 0$. If $-f^*(K_{\bar{X}}) \cdot G = 0$, then $G_0 \cdot M_0 = (G_0 \cdot M_0)_{P_1} = s$ and $G_0 \cdot F_1 = (G_0 \cdot F_1)_{P_1} = 2s$; that is, $G_0 \sim 2sM_0 + sF_1$.

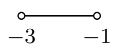
Note that G is disjoint from F_2'. Then $G \cdot C = 2s - G \cdot F_2' = 2s$. However, this would imply that G_0 has multiplicity $2s$ at the point $\Phi(C)$, and thus $s = G_0 \cdot M_0 \geq 2s$, a contradiction again.

Therefore, $-K_{\bar{X}} \cdot \bar{G} = f^*(K_{\bar{X}}) \cdot G > 0$ for every irreducible curve $\bar{G}$ on $\bar{X}$. Since $(-K_{\bar{X}})^2 > 0$, by Nakai-Moishezon criterion, $-K_{\bar{X}}$ is ample for all the 29 configurations listed in Figure 1.6. We have completed the proof of Lemma 1.12. □

1.6 The List of Weighted Dual Graphs of $C + D$

I (a).

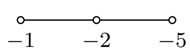
I (b):



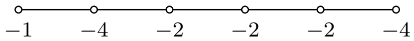
II (a):

II (b):

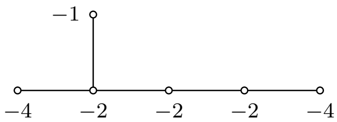
III:



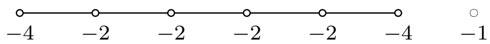
V ($n = 5$) (a):



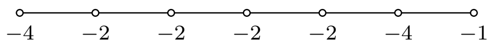
V ($n = 5$) (b):



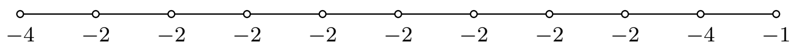
V ($n = 6$) (a):



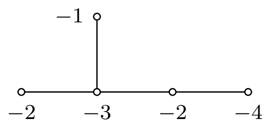
V ($n = 6$) (b):



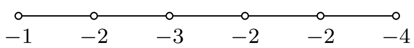
V ($n = 10$):



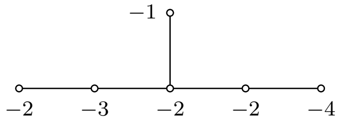
VI ($n = 4$):



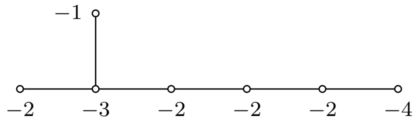
VI ($n = 5$) (a):



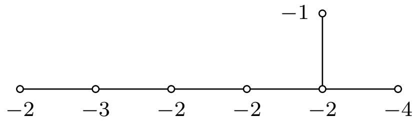
VI ($n = 5$) (b):



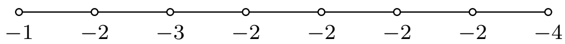
VI ($n = 6$) (a):



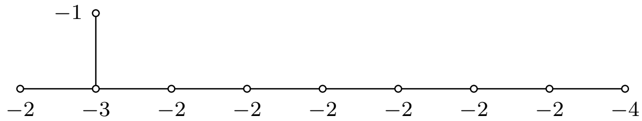
VI ($n = 6$) (b):



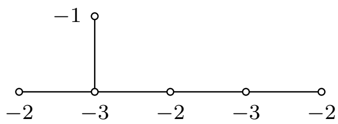
VI ($n = 7$):



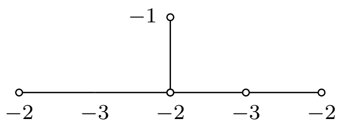
VI ($n = 9$):



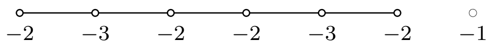
VII ($n = 5$) (a):



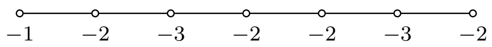
VII ($n = 5$) (b):



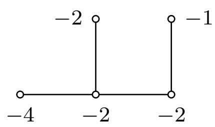
VII ($n = 6$) (a):



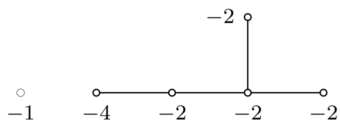
VII ($n = 6$) (b):



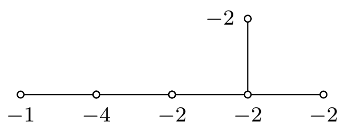
VIII ($n = 4$):



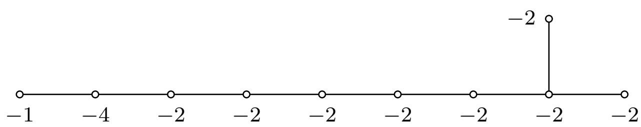
VIII ($n = 5$) (a):



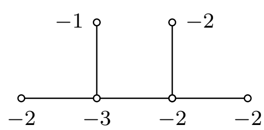
VIII ($n = 5$) (b):



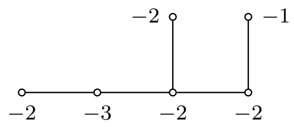
VIII ($n = 9$)



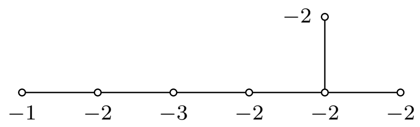
IX ($n = 5$) (a):



IX ($n = 5$) (b):



IX ($n = 6$):



IX ($n = 8$):

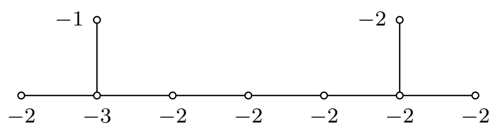


Figure 1.6: Weighted Dual graphs of $C + D$

Chapter 2

Log Enriques Surfaces of Rank 18

2.1 Introduction

A normal projective surface Z with at worst quotient singularities is called a *logarithmic* (abbr. *log*) *Enriques surface* if its canonical Weil divisor K_Z is numerically equivalent to zero, and if its irregularity $\dim H^1(Z, \mathcal{O}_Z) = 0$. By the abundance for surfaces, $K_Z \sim_{\mathbb{Q}} 0$.

Let Z be a log Enriques surface and define

$$I := I(Z) = \min\{n \in \mathbb{Z}^+ \mid \mathcal{O}_Z(nK_Z) \simeq \mathcal{O}_Z\}$$

to be the *canonical index* of Z. The *canonical cover* of Z is defined as

$$\pi : \bar{S} := \mathrm{Spec}_{\mathcal{O}_Z}\left(\bigoplus_{j=0}^{I-1} \mathcal{O}_Z(-jK_Z)\right) \to Z.$$

This is a Galois $\mathbb{Z}/I\mathbb{Z}$-cover. So $\bar{S}/(\mathbb{Z}/I\mathbb{Z}) = Z$.

Note that a log Enriques surface is irrational if and only if it is a K3 or Enriques surface with at worst Du Val singularities (cf. [41, Proposition 1.3]). More precisely, a log Enriques surface of index one is a K3 surface with at worst Du Val singularities, and a log Enriques surface of index two is an Enriques surface with at worst Du Val singularities or a rational surface. Therefore, the log Enriques surfaces can be viewed as generalizations of K3 surfaces and Enriques surfaces.

More results about the canonical indices are studied in by Zhang in [41] and [42].

If a log Enriques surface Z has Du Val singularities, let $\widetilde{Z} \to Z$ be the partial minimal resolution of all Du Val singularities of Z, then $\widetilde{Z}$ is again a log Enriques surface of the same canonical index as Z. Therefore, we assume throughout this chapter that Z has no Du Val singularities; otherwise we consider $\widetilde{Z}$ instead.

By the definition of the canonical cover and the classification result of surfaces, we have the following (cf. [41]).

1. $\bar{S}$ has at worst Du Val singularities, and its canonical divisor $K_{\bar{S}}$ is linearly equivalent to zero. So $\bar{S}$ is either an abelian surface or a projective K3 surface with at worst Du Val singularities.

2. $\pi : \bar{S} \to Z$ is a finite, cyclic Galois cover of degree $I = I(Z)$, and it is étale over $Z \backslash \operatorname{Sing} Z$.

3. $\operatorname{Gal}(\bar{S}/Z) \simeq \mathbb{Z}/I\mathbb{Z}$ acts faithfully on $H^0(\mathcal{O}_{\bar{S}}(K_{\bar{S}}))$. In other words, there is a generator g of $\operatorname{Gal}(\bar{S}/Z)$ such that $g^*\omega_{\bar{S}} = \zeta_I \omega_{\bar{S}}$, where ζ_I is the Ith primitive root of unity and $\omega_{\bar{S}}$ is a nowhere vanishing regular 2-form on $\bar{S}$.

Suppose $\operatorname{Sing} \bar{S} \neq \emptyset$. Let $\nu : S \to \bar{S}$ be the minimal resolution of $\bar{S}$, and Δ_S the exceptional divisor of ν. Then Δ_S is a disconnected sum of divisors of Dynkin's type:

$$(\oplus A_\alpha) \oplus (\oplus D_\beta) \oplus (\oplus E_\gamma)$$

Note that S is a K3 surface. The Chern map $c_1 : \operatorname{Pic}(S) \to H^2(S, \mathbb{Z})$ is injective. So $\operatorname{Pic}(S)$ is mapped isomorphically onto the Neron-Severi group $\operatorname{NS}(S)$. We can therefore define the *rank* of Δ_S to be the rank of the sublattice of the Néron Severi lattice $\operatorname{NS}(S) \simeq \operatorname{Pic}(S)$ generated by the irreducible components of Δ_S. In other words,

$$\operatorname{rank} \Delta_S = \sum \alpha + \sum \beta + \sum \gamma.$$

Moreover, let $\rho(S) := \operatorname{rank} \operatorname{Pic}(S)$ be the Picard number of S, then

$$\operatorname{rank} \Delta_S \leq \rho(S) - 1 \leq 20 - 1 = 19.$$

Since S is uniquely determined up to isomorphism, by abuse of language we also say Z is of type $(\oplus A_\alpha) \oplus (\oplus D_\beta) \oplus (\oplus E_\gamma)$, and call $\operatorname{rank} \Delta_S$ the *rank* of Z.

A rational log Enriques surface Z is called *extremal* if it is of rank 19, the maximal possible value 19. The extremal log Enriques surfaces are completely classified by Oguiso and Zhang in [33]. In [32], they determined the isomorphism classes of rational log Enriques surfaces of type A_{18} and D_{18}.

In this chapter, we are going to classify all the rational log Enriques surfaces of rank 18 by proving the following theorem.

Theorem 2. *Let Z be a rational log Enriques surfaces of rank* 18 *without Du Val* *singularities. Let $\bar{S} \to Z$ be the canonical cover, and $S \to \bar{S}$ the minimal resolution with exceptional divisor Δ_S. Then we have the following assertions.*

1) *The canonical index $I(Z) = 2$, 3 or 4.*

2) *If $I(Z) = 2$, then $(S, g) \simeq (S_2, g_2)$, and Δ_S is of one of the following 5 types:*

$$A_1 \oplus A_{17}, \quad A_3 \oplus A_{15}, \quad A_5 \oplus A_{13}, \quad A_7 \oplus A_{11}, \quad A_9 \oplus A_9.$$

   Moreover, all of them are realizable.

3) *If $I(Z) = 3$, then $(S, g) \simeq (S_3, g_3)$, and Δ_S is of one of the 48 possible types in Table 2.1, and from which 40 types have been realized.*

4) *If $I(Z) = 4$, then $(S, g^2) \simeq (S_2, g_2)$, and Δ_S is of one of the following 3 types:*

$$A_1 \oplus A_{17}, \quad A_5 \oplus A_{13}, \quad A_9 \oplus A_9.$$

   Moreover, all of them are realizable.

5) *For each of the possible cases in (2) and (3), every irreducible curve in Δ_S is g-stable, and the action of g on Δ is uniquely determined, which are given in Table 2.2 and 2.1, respectively.*

Here (S_2, g_2) (Definition 2.6) and (S_3, g_3) (Definition 2.3) are the Shioda-Inose's pairs of discriminants 4 and 3, respectively.

2.2 Preliminaries

Definition 2.1. Let Z be a normal projective surface defined over the complex number field $\mathbb{C}$. It is called a *log Enriques surface* if

1) Z has at worst quotient singularities,

2) IK_Z is linearly equivalent to zero for some positive integer I, and

3) the irregularity $q(Z) := \dim H^1(Z, \mathcal{O}_Z) = 0$.

The smallest positive integer I in condition (2) is called the *canonical index* of Z.

We will use the following notations in Sections 2.3–2.4.

1. For each $I \in \mathbb{Z}^+$, $\zeta_I = \exp(2\pi\sqrt{-1}/I)$, a primitive Ith root of unity.

2. Let X be a variety, and G an automorphism group on X. For each $g \in X$, denote the fixed locus by $X^g = \{x \in X \mid g(x) = x\}$. Set $X^{[G]} = \bigcup_{g\in G\setminus\{\mathrm{id}\}} X^g$.

3. Let S be a surface and g an automorphism on S. A curve C on S is called *g-stable* if $g(C) = C$, and it is called *g-fixed* if $g(x) = x$ for every $x \in C$. A point $x \in S$ is an *isolated g-fixed point* if $g(x) = x$ and it is not contained in any g-fixed curve.

2.3 Log Enriques Surfaces from Shioda-Inose's Pairs

In this section, we assume that Z is a rational log Enriques surface of rank 18 and canonical index I without Du Val singularities. Let $\pi : \bar{S} \to Z$ be the canonical cover of Z, and $\nu : S \to \bar{S}$ the minimal resolution of $\bar{S}$ with exceptional divisor Δ_S. Then

$$20 \geq \rho(S) \geq \operatorname{rank} \Delta_S + 1 = 19.$$

Recall that S is a K3 surface. Let T_S denote the transcendental lattice of S, i.e., the orthogonal complement of $\operatorname{Pic}(S)$ in $H^2(S, \mathbb{Z})$. Then

$$\operatorname{rank} T_S = \dim H^2(S, \mathbb{Z}) - \rho(S) = 22 - \rho(S) = 2 \text{ or } 3.$$

Let g be the automorphism on S induced by a generator of $\operatorname{Gal}(\bar{S}/Z)$, and ω_S a nowhere vanishing holomorphic 2-form on S. Then $g^*\omega_S = \zeta_I \omega_S$. Note that $\omega_S \in T_S \otimes \mathbb{C}$. So ζ_I is an eigenvalue of g^* acting on T_S. Therefore, $\varphi(I) \leq \operatorname{rank} T_S \leq 3$, where φ is Euler's phi function. It follows that

Lemma 2.2. *The canonical index* $I(Z) = 2, 3, 4$ *or* 6.

We have indicated that all the realizable rational log Enriques surfaces listed in Theorem 2 can be constructed from the Shioda-Inose's pairs (S_2, g_2) or (S_3, g_3) (cf. [34]). Precisely, if $I(Z) = 2$, then $(S, g) \simeq (S_2, g_2)$; if $I(Z) = 3$, then $(S, g) \simeq (S_3, g_3)$; if $I(Z) = 4$, then $(S, g^2) \simeq (S_2, g_2)$; we will also show that $I \neq 6$.

Definition 2.3. Let $\zeta_3 := \exp(2\pi\sqrt{-1}/3)$, and $E_{\zeta_3} := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_3)$ the elliptic curve of period ζ_3. Let $\bar{S}_3 := E_{\zeta_3}^2/\langle \operatorname{diag}(\zeta_3, \zeta_3^2) \rangle$ be the quotient surface, and $S_3 \to \bar{S}_3$ the minimal resolution of $\bar{S}_3$.

Let g_3 be the automorphism of S_3 induced by the action $\mathrm{diag}(\zeta_3, 1)$ on $E_{\zeta_3}^2$. Then (S_3, g_3) is called the *Shioda-Inose's pair of discriminant* 3.

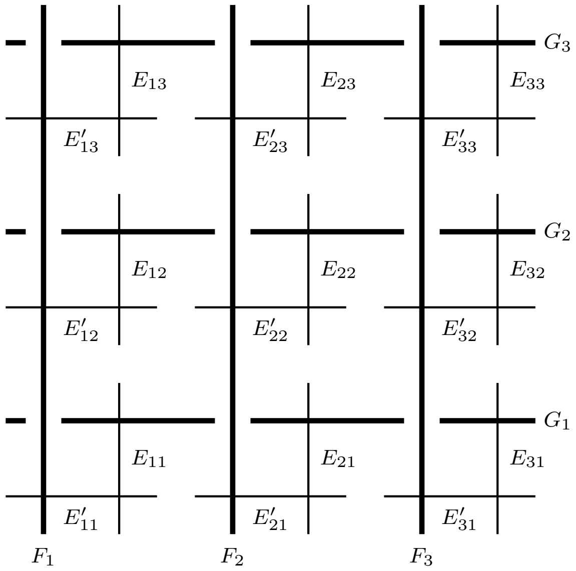


Figure 2.1: (S_3, g_3)

It is proved in [31] and [33] that

Proposition 2.4. *Let (S_3, g_3) be the Shioda-Inose's pair of discriminant 3. Then*

1) *S_3 contains 24 rational curves: F_1, F_2, F_3 coming from $(E_{\zeta_3})^{\zeta_3} \times E_{\zeta_3}$; G_1, G_2, G_3 coming from $E_{\zeta_3} \times (E_{\zeta_3})^{\zeta_3}$; and E_{ij}, E'_{ij} $(i, j = 1, 2, 3)$ the exceptional curves arising from the 9 Du Val singular points of $\bar{S}_3$ (Figure. 2.1);*

2) *$g_3^* \omega_{S_3} = \zeta_3 \omega_3$, where ω_{S_3} is a nowhere vanishing holomorphic 2-form on S_3, and $g_3^*|_{\mathrm{Pic}(S_3)} = \mathrm{id}$; so each of the 24 curves is g_3-stable;*

3) *$S_3^{g_3} = (\coprod_{i=1}^3 F_i) \coprod (\coprod_{j=1}^3 G_j) \coprod (\coprod_{i,j=1}^3 \{P_{ij}\})$, where $\{P_{ij}\} = E_{ij} \cap E'_{ij}$;*

4) *$g_3 \circ \varphi = \varphi \circ g_3$ for all $\varphi \in \mathrm{Aut}(S_3)$.*

Proposition 2.5. *Let (S, g) be a pair of a smooth K3 surface S and an automorphism of g on S. Assume that*

1) $g^3 = \mathrm{id}$, *the identity on S;*

2) $g^*\omega_S = \zeta_3\omega_S$, *where ω_S is a nowhere vanishing holomorphic 2-form on S;*

3) S^g *consists of only rational curves and isolated points;*

4) S^g *contains at least 6 rational curves.*

Then $(S, g) \simeq (S_3, g_3)$. Moreover, S^g consists of exactly 6 rational curves and 9 isolated points.

Definition 2.6. Let $E_{\zeta_4} := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ be the elliptic curve of period $\zeta_4 = \sqrt{-1}$. Let $\bar{S}_2 := E_{\zeta_4}^2/\langle \mathrm{diag}(\zeta_4, \zeta_4^3)\rangle$ be the quotient surface and $S_2 \to \bar{S}_2$ the minimal resolution of $\bar{S}_2$.

Let g_2 be the involution of S_2 induced by the action $\mathrm{diag}(-1, 1)$ on $E_{\zeta_4}^2$. Then (S_2, g_2) is called the *Shioda-Inose's pair of discriminant* 4.

It is also proved in [31] and [33] that

Proposition 2.7. *Let (S_2, g_2) be the Shioda-Inose's pair of discriminant 4. Then*

1) *S_2 contains 24 rational curves: F_1, F_2, F_3 coming from $(E_{\zeta_4})^{[\langle\zeta_4\rangle]} \times E_{\zeta_4}$; G_1, G_2, G_3 coming from $E_{\zeta_4} \times (E_{\zeta_4})^{[\langle\zeta_4\rangle]}$; and $E'_{ij} + H_{ij} + E_{ij}$, $i, j \in \{1, 3\}$, the exceptional curves arising from the 4 Du Val singular points of Dynkin type A_3; and*

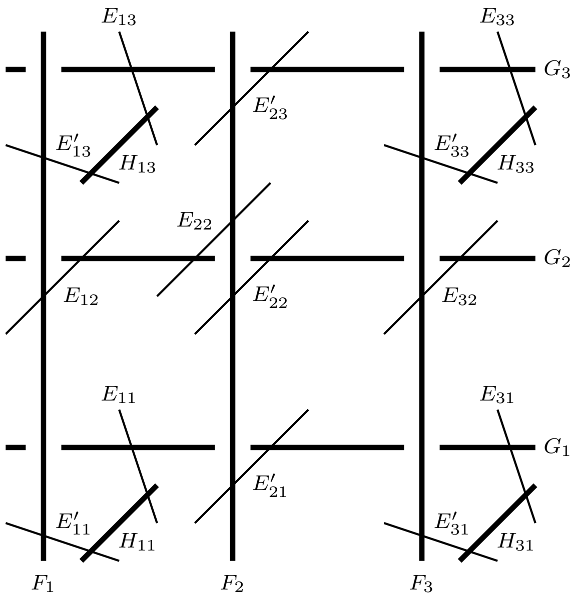


Figure 2.2: (S_2, g_2)

$E_{12}, E_{22}, E_{32}, E'_{21}, E'_{22}, E'_{23}$, *the exceptional curves arising from the* 6 *Du Val singular points of Dynkin type* A_1 (*Figure. 2.2*);

2) $g_2^*\omega_{S_2} = -\omega_{S_2}$, *where* ω_{S_2} *is a nowhere vanishing holomorphic* 2*-form on* S_2, *and* $g_2^*|_{\mathrm{Pic}(S)} = \mathrm{id}$*; so each of the* 24 *curves is* g_2*-stable;*

3) $S_2^{g_2} = (\coprod_{i=1}^3 F_i) \coprod (\coprod_{j=1}^3 G_j) \coprod (\coprod_{i,j\in\{1,3\}} H_{ij})$*;*

4) $g_2 \circ \varphi = \varphi \circ g_2$ *for all* $\varphi \in \mathrm{Aut}(S_2)$.

Proposition 2.8. *Let* (S, g) *be a pair of a smooth K3 surface* S *and an automorphism* g *of* S. *Assume that*

1) $g^2 = \mathrm{id}$, *the identity on* S*;*

2) $g^*\omega_S = -\omega_S$, *where* ω_S *is a nowhere vanishing holomorphic* 2*-form on* S*;*

3) *S^g consists of only rational curves;*

4) *S^g contains at least 10 rational curves.*

Then $(S, g) \simeq (S_2, g_2)$. Moreover, S^g consists of exactly 10 rational curves.

2.4 The Classification

In this section, we assume that Z is a rational log Enriques surface of rank 18 without Du Val singularities. Let $\pi : \bar{S} \to Z$ be the canonical cover, and $\nu : S \to \bar{S}$ the minimal resolution with exceptional divisor $\Delta := \Delta_S$.

Since the canonical cover $\bar{S} \to Z$ is unramified in codimension one, every curve in $S^{[\langle g \rangle]}$ is contained in Δ. In particular, $S^{[\langle g \rangle]}$ consists of only smooth rational curves and a finite number of isolated points, and Δ is g-stable.

In general, let S be a K3 surface, and g an automorphism of S of order n. Let T_S be its transcendental lattice. Note that g induces actions g^* on $\mathrm{Pic}(S) \otimes \mathbb{C}$ and on $T_S \otimes \mathbb{C}$. Since $g^n = \mathrm{id}$, these actions are diagonalizable and every eigenvalue of g^* is an nth root of unity, say ζ_n^i for some $0 \leq i < n$.

Since g^* is well-defined on $\mathrm{Pic}(S)$ and T_S, the number of eigenvalues ζ_n^i of $g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}}$ and $g^*|_{T_S\otimes\mathbb{C}}$ equals to that of the conjugate eigenvalues $\bar{\zeta}_n^i$, respectively. By noting that $\dim H^2(S, \mathbb{C}) = 22$, we have the following lemma:

Lemma 2.9 ([31, Lemma 2.0])**.** *With the notations above, let t_0 and r_0 be the rank* *of the invariant lattices* $(\mathrm{Pic}(S))^{g^*}$ *and* $(T_S)^{g^*}$*, respectively. Let* I_s *denote the identity matrix of size* s.

1) *If* $n = 2k+1$ *is odd, then* $\rho(S) = t_0 + 2\sum_{i=1}^{k} t_i$ *and*

$$g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}} = \mathrm{diag}(I_{t_0}, \zeta_n I_{t_1}, \bar{\zeta}_n I_{t_1}, \zeta_n^2 I_{t_2}, \bar{\zeta}_n^2 I_{t_2}, \dots, \zeta_n^k I_{t_k}, \bar{\zeta}_n^k I_{t_k}),$$

$$g^*|_{T_S\otimes\mathbb{C}} = \mathrm{diag}(I_{r_0}, \zeta_n I_{r_1}, \bar{\zeta}_n I_{r_1}, \zeta_n^2 I_{r_2}, \bar{\zeta}_n^2 I_{r_2}, \dots, \zeta_n^k I_{r_k}, \bar{\zeta}_n^k I_{r_k}),$$

and $t_0 + r_0 + 2\sum_{i=1}^{k} t_i + 2\sum_{i=1}^{k} r_i = 22$.

2) *If* $n = 2k$ *is even, then* $\rho(S) = t_0 + 2\sum_{i=1}^{k-1} t_i + t_k$ *and*

$$g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}} = \mathrm{diag}(I_{t_0}, \zeta_n I_{t_1}, \bar{\zeta}_n I_{t_1}, \zeta_n^2 I_{t_2}, \bar{\zeta}_n^2 I_{t_2}, \dots, \zeta_n^{k-1} I_{t_{k-1}}, \bar{\zeta}_n^{k-1} I_{t_{k-1}}, -I_{t_k}),$$

$$g^*|_{T_S\otimes\mathbb{C}} = \mathrm{diag}(I_{r_0}, \zeta_n I_{r_1}, \bar{\zeta}_n I_{r_1}, \zeta_n^2 I_{r_2}, \bar{\zeta}_n^2 I_{r_2}, \dots, \zeta_n^{k-1} I_{r_{k-1}}, \bar{\zeta}_n^{k-1} I_{r_{k-1}}, -I_{r_k}),$$

and $t_0 + r_0 + 2\sum_{i=1}^{k-1} t_i + 2\sum_{i=1}^{k} r_i + t_k + r_k = 22$.

2.4.1 Classification When $I = 3$

Let (S, g) be a pair of smooth K3 surface S and an automorphism g of S. We assume that $g^*\omega_S = \zeta_3\omega_S$ for a nowhere vanishing holomorphic 2-form ω_S on S.

Let P be an isolated g-fixed point on S. Then g^* can be written as $\mathrm{diag}(\zeta_3^a, \zeta_3^b)$ for some $a, b \in \{1, 2\}$ with $a + b \equiv 1 \pmod 3$ under some appropriate local coordinates around P because $g^*\omega_S = \zeta_3\omega_S$. We see that $a = b = 2$ and the action is $\mathrm{diag}(\zeta_3^2, \zeta_3^2)$.

If C is a g-fixed irreducible curve and $Q \in C$, then it also follows from $g^*\omega_S = \zeta_3\omega_S$ that g^* can be written as $\mathrm{diag}(1, \zeta_3)$ under some appropriate local coordinates around Q. In particular, the g-fixed curves are smooth and mutually disjoint.

We need to use the following lemma in the classification for $I = 3$.

Lemma 2.10 ("Three Go" Lemma, [31, Lemma 2.2]). *Let (S, g) be a pair of smooth K3 surface S and an automorphism g of S. Assume that $g^3 = \mathrm{id}$ and $g^*\omega_S = \zeta_3\omega_S$.*

1) *Let $C_1 - C_2 - C_3$ be a linear chain of g-stable smooth rational curves. Then exactly one of C_i is g-fixed.*

2) *Let C be a g-stable but not g-fixed smooth rational curve. Then there is a unique g-fixed curve D such that $C \cdot D = 1$.*

3) *Let M and N be the number of smooth rational curves and the number of isolated points in S^g, respectively. Then $M - N = 3$.*

Suppose $I(Z) = 3$. Then the associated pair (S, g) satisfies the conditions in Lemma 2.10. We first determine a possible list of the Dynkin's types of Δ.

Proposition 2.11. *With the notations as in Theorem 2, suppose $I(Z) = 3$. Then $(S, g) \simeq (S_3, g_3)$, the Shioda-Inose's pair of discriminant 3. Moreover, Δ is of one of the following 13 types:*

I. A_{18};

II. D_{18};

III. $A_{3m} \oplus A_{3n}, \quad m + n = 6$;

IV. $D_{3m} \oplus A_{3n}, \quad m + n = 6$;

V. $D_{3m} \oplus D_{3n}, \quad m + n = 6$;

VI. $D_{3m+1} \oplus A_{3n-1}, \quad m+n=6$;

VII. $A_{3m} \oplus A_{3n} \oplus A_{3r}, \quad m+n+r=6$;

VIII. $D_6 \oplus D_6 \oplus D_6$,

IX. $A_{3m} \oplus D_{3n} \oplus D_{3r}, \quad m+n+r=6$;

X. $A_{3m} \oplus A_{3n} \oplus D_{3r}, \quad m+n+r=6$;

XI. $D_{3m+1} \oplus A_{3n} \oplus A_{3r-1}, \quad m+n+r=6$;

XII. $D_{3m+1} \oplus D_{3n+1} \oplus A_{3r-2}, \quad m+n+r=6$;

XIII. $D_{3m+1} \oplus D_{3n} \oplus A_{3r-1}, \quad m+n+r=6$.

Proof. Let Δ_i be a connected component of Δ.

Step 1: Δ_i is g-stable.

If Δ_i is not g-stable, then its image in Z would be a Du Val singular point since $I(Z)=3$ is a prime. However, we have assumed that Z has no Du Val singularities.

Step 2: $\Delta_i = A_n$ or D_n.

Suppose there is a $\Delta_i = E_n$ for some n. Let C be the center of Δ_i, and C_1, C_2, C_3 the rational curves in Δ_i which intersect C. Suppose C_1 is the twig of length one.

By the uniqueness of C and C_1, they are g-stable. If C is not g-fixed, then $\Delta_i = E_6$ and g switches the other two twigs, which contradicts $g^3 = \mathrm{id}$. If C is g-fixed, then each irreducible curve in Δ_i is g-stable. Let $C_2 - C_2'$ be a twig of Δ_i. Then C_2' is not g-fixed and it does not intersect with any g-fixed curve, which contradicts

Lemma 2.10.

Step 3. Every irreducible curve in Δ_i is g-stable.

i) Let $\Delta_i = A_n$. Write the irreducible curves in Δ_i as a chain $C_1 - C_2 - \cdots - C_n$. For $n > 1$, if C_1 is not g-stable, we must have $g(C_1) = C_n$ and $g(C_n) = g(C_1)$, and this contradicts $g^3 = \mathrm{id}$.

ii) Let $\Delta_i = D_n$. Then by the uniqueness its center C is g-stable. Let C_1 and C_2 be twigs of length one, and C_3 the curve of another twig which intersects C.

Suppose $n > 4$. Then every irreducible component in the longest twig shall be g-stable. If C_1 is not g-stable, then $g(C_1) = C_2$ and $g(C_2) = C_1$, which contradicts $g^3 = \mathrm{id}$. Thus, every irreducible curve in Δ_i is g-stable.

Suppose $n = 4$. If C_1 is not g-stable, we must have $g(C_1) = C_2$, $g(C_2) = C_3$ and $g(C_3) = g(C_1)$. In particular, C is not g-fixed, and it does not intersect with any g-fixed curve. This contradicts Lemma 2.10. Therefore, C_1 is g-stable. We see similarly as in the case $n > 4$ that C_2 and C_3 are both g-stable.

Step 4. The g-fixed curves of Δ_i are described as follows.

We use "f" to denote g-fixed curves, and "s" to denote g-stable but not g-fixed curves in Δ_i. k is the number of g-fixed curves in Δ_i.

i) Suppose $\Delta_i = A_n$.

a) $n = 3k - 2$:

$$f \text{ - } s \text{ - } s \text{ - } f \text{ - } s \text{ - } \cdots \text{ - } s \text{ - } s \text{ - } f$$

b) $n = 3k-1$:

$$f - s - s - f - s - \cdots - s - f - s$$

c) $n = 3k$:

$$s - f - s - s - f - \cdots - s - f - s$$

ii) Suppose $\Delta_i = D_n$.

a) $n = 3k$:

$$\begin{array}{ccccccccccccccccc} & & s & & & & & & & & & & & & & & \\ & & | & & & & & & & & & & & & & & \\ s & - & f & - & s & - & s & - & f & - & \cdots & - & s & - & s & - & f \end{array}$$

b) $n = 3k+1$:

$$\begin{array}{ccccccccccccccccc} & & s & & & & & & & & & & & & & & \\ & & | & & & & & & & & & & & & & & \\ s & - & f & - & s & - & s & - & f & - & \cdots & - & s & - & f & - & s \end{array}$$

The case $\Delta_i = A_n$ follows from Lemma 2.10. Suppose $\Delta_i = D_n$. Then by Step 3, the center C is g-fixed. So in the longest twig $C_3 - C_4 - \cdots - C_{n-1}$ of Δ_i, by induction, C_{3j+2} are g-fixed and others are not. If $n = 3k+2$ for some k, then C_{n-2} and C_{n-1} are not g-fixed, and C_{n-1} does not intersect with any g-fixed curve, contradicting to Lemma 2.10. Therefore, $n \not\equiv 2 \pmod 3$.

Step 5. $(S, g) \simeq (S_3, g_3)$.

Let M be the number of isolated g-fixed points and N the number of g-fixed curves in Δ. We can decompose

$$\Delta = \bigoplus_{i=1}^{a} D_{3\ell_i+1} \oplus \bigoplus_{i=1}^{b} D_{3m_i} \oplus \bigoplus_{i=1}^{c} A_{3p_i} \oplus \bigoplus_{i=1}^{d} A_{3q_i-1} \oplus \bigoplus_{i=1}^{e} A_{3r_i-2}.$$

Then

$$\begin{aligned} N &= \sum_{i=1}^{a} \ell_i + \sum_{i=1}^{b} m_i + \sum_{i=1}^{c} p_i + \sum_{i=1}^{d} q_i + \sum_{i=1}^{e} r_i, \\ M &\geq \sum_{i=1}^{a}(\ell_i + 2) + \sum_{i=1}^{b}(m_i + 1) + \sum_{i=1}^{c}(p_i + 1) + \sum_{i=1}^{d} q_i + \sum_{i=1}^{e}(r_i - 1) \\ &= N + (2a + b + c - e). \end{aligned}$$

Thus, by Lemma 2.10, $3 = M - N \geq 2a + b + c - e$.

Recall that

$$\begin{aligned} \operatorname{rank} \Delta = 18 &= \sum_{i=1}^{a}(3\ell_i + 1) + \sum_{i=1}^{b} 3m_i + \sum_{i=1}^{c} 3p_i + \sum_{i=1}^{d}(3q_i - 1) + \sum_{i=1}^{e}(3r_i - 2) \\ &= 3N + a - d - 2e. \end{aligned}$$

Or equivalently, $N = 6 + \dfrac{-a + d + 2e}{3}$. If $N \leq 5$, then $a \geq d + 2e + 3$, and we would have

$$3 \geq 2a + b + c - e \geq 2(d + 2e + 3) + b + c - e = b + c + 2d + 3e + 6 \geq 6.$$

Therefore, $N \geq 6$; and hence by Proposition 2.5, $N = 6$ and $M = 9$. Furthermore, we have $(S, g) \simeq (S_3, g_3)$.

Step 6. Determine the Dynkin's type of Δ.

Solving the system

$$d + 2e = a \quad \text{and} \quad 2a + b + c - e \leq 3,$$

we have 13 nonnegative integer solutions. So there are 13 types of Δ as listed in Proposition 2.11. $\square$

To be more precise, we list all the 48 possible types of Δ in Table 2.1 at the end of this chapter. Note that in Steps 3 and 4, we proved that each irreducible curve in Δ g-stable, and the action of g on Δ is uniquely determined, which is also included in Table 2.1. The case $I = 3$ for Theorem 2 (5) is proved.

If Δ can be obtained from the 24 g-stable rational curves in S_3 (Figure 2.1) which contains the 6 g-fixed curves and satisfies the condition in the proof of Proposition 2.11 Step 4, let $S_3 \to \bar{S}$ be the contraction of Δ, then the automorphism g_3 on S_3 induces an automorphism on $\bar{S}$. We see that $Z = \bar{S}/\langle g_3 \rangle$ is a required log Enriques surface of type Δ. By verification, 40 cases are realizable. The detailed list is given in Table 2.1(A).

Thus, we have completed the proof of Theorem 2 (3).

Unfortunately, the remaining 8 cases are not realizable by the the 24 curves on S_3, which are given in Table 2.1 (B). We are unable to determine their realizability.

2.4.2 Classification When $I = 2$

Let (S, g) be a pair of a smooth K3 surface S and an automorphism g of S. We assume that $g^*\omega_S = -\omega_S$ for a nowhere vanishing holomorphic 2-form ω_S on S.

If $P \in S$ is an isolated g-fixed point, then g^* can be written as $\mathrm{diag}(-1, -1)$ under some appropriate local coordinates around P. However, this contradicts the assumption that $g^*\omega_S = -\omega_S$. So S has no isolated g-fixed point.

Let C be a g-fixed irreducible curve and let $Q \in C$. Then g^* can be written as $\mathrm{diag}(1, -1)$ under some appropriate local coordinates around Q. So the g-fixed curves are smooth and mutually disjoint.

We need to use the following lemma in the classification.

Lemma 2.12 ("Two Go" Lemma, [31, Lemma 3.2]). *Let (S, g) be a pair of smooth K3 surface and an automorphism g of S. Assume that $g^2 = \mathrm{id}$ and $g^*\omega_S = -\omega_S$.*

1) *If $C_1 - C_2$ is a linear chain of g-stable smooth rational curves, then exactly one of C_i is g-fixed.*

2) *If C_1 and C_2 are g-stable but not g-fixed smooth rational curves, then $C_1 \cdot C_2$ is even.*

3) *If C is a g-stable but not g-fixed smooth rational curve, then C has exactly 2 g-fixed points.*

Suppose $I(Z) = 2$. Then the associated pair satisfies the conditions in Lemma 2.12. We can now determine the possible Dynkin's types of Δ.

Proposition 2.13. *With the notations as in Theorem 2. Suppose $I = 2$. Then $(S, g) \simeq (S_2, g_2)$, the Shioda-Inose's pair of discriminant 4. Moreover, Δ is of the type $A_{2m-1} \oplus A_{2n-1}$, where $m + n = 10$.*

Proof. Since $I = 2$ is a prime, each connected component Δ_i of Δ must be g-stable because Z is assumed to have no Du Val singular points.

Step 1. $\Delta_i = A_n$.

Suppose $\Delta_i = D_n$ or E_n. Let C be the center of Δ_i. Then C meets exactly 3 smooth rational curves in Δ_i, say C_1, C_2, C_3. By uniqueness, C is g-stable, and $g(\{C_1, C_2, C_3\}) = \{C_1, C_2, C_3\}$.

If every C_j is g-stable, then C has at least 3 g-fixed points, and it is g-fixed. Hence, C_j are not g-fixed. On the other hand, each C_j contains two g-fixed points, and one of them is not in C. There would be another g-fixed curve C_j' in Δ_i which intersects C_j, $j = 1, 2, 3$, a contradiction.

Suppose C_1 is not g-stable, say $g(C_1) = C_2$. Then $g(C_2) = C_1$ and C is not g-fixed. Since C_3 is g-stable, by Lemma 2.12 it is also g-fixed. However, one of the two g-fixed points on C is not contained in C_3, so C should intersect with another g-fixed curve in Δ_i, a contradiction again.

Therefore, we can express $\Delta_i = A_n$ as a linear chain of smooth rational curves: $C_1 - C_2 - \cdots - C_n$.

Step 2. Each C_j is g-stable.

Suppose $g(C_1) \neq C_1$. Then $g(C_1) = C_n$, and consequently $g(C_j) = C_{n-j}$ for all j. There are two cases.

i) If $m = 2k$, let $\{P\} = C_k \cap C_{k+1}$, then P would be an isolated g-fixed point, absurd.

ii) If $m = 2k - 1$, then C_k is g-stable, and there would be a g-fixed curve which intersects C_k. But Δ_i contains no g-fixed curve, a contradiction.

Therefore, $g(C_1) = C_1$ and it follows that each C_j is g-stable.

Step 3. $\Delta_i = A_{2m-1}$.

Note that each g-stable but not g-fixed curve must intersect g-fixed curves at two points. So C_1 is g-fixed and C_2 is not. Consequently, each C_{2j-1} is g-fixed and C_{2j} is not. With the same reason, C_n must be g-fixed. So n is odd. Therefore, $\Delta_i = A_n$ has the form

$$f - s - f - s - f - \cdots - f - s - f$$

where "f" denotes the g-fixed curves and "s" denotes the g-stable but not g-fixed curves in Δ_i.

Step 4. Determine the Dynkin type of Δ.

Decompose $\Delta = \bigoplus_{i=1}^{r} A_{2n_i-1}$. Recall that every smooth rational g-fixed curve in S is contained in Δ. Let N be the number of smooth rational g-fixed curves in S. Then $N = \sum_{i=1}^{r} n_i$ and

$$18 = \operatorname{rank} \Delta = \sum_{i=1}^{r} (2n_i - 1) = 2N - r.$$

So we have

$$N = \frac{18 + r}{2} > 9.$$

Then $N \geq 10$. It follows from Proposition 2.8 that $N = 10$ and $(S, g) \simeq (S_2, g_2)$. Moreover, $r = 2$. This completes the proof. $\square$

We have the following configurations for Δ:

$$A_1 \oplus A_{17}, \quad A_3 \oplus A_{15}, \quad A_5 \oplus A_{13}, \quad A_7 \oplus A_{11}, \quad A_9 \oplus A_9.$$

Similarly as in the case when $I = 3$, if $S_2^g \subseteq \Delta$ and the divisor Δ can be obtained from the 24 smooth rational curves in S_2 (Figure 2.2) which satisfies the conditions of Step 3 in the proof of Proposition 2.13, let $S_2 \to \bar{S}$ be the contraction of Δ, then the automorphism g_2 on S_2 induces an automorphism on $\bar{S}$, and $Z := \bar{S}/\langle g_2 \rangle$ is a required log Enriques surface of Dynkin's type Δ.

We can easily verify that these 5 cases are all realizable (Table 2.2). Theorem 2 (2) is proved. By noting the results in Steps 2 and 3 in the proof of Proposition 2.13, Theorem 2 (5) for case $I = 2$ is also proved.

2.4.3 Classification When $I = 4$

Let (S, g) be a pair of a smooth K3 surface S and an automorphism g of S. Assume that $g^4 = \mathrm{id}$ and $g^*\omega_S = i\omega_S$ for a nowhere vanishing holomorphic 2-form on S, where $i = \sqrt{-1}$.

Let P be an isolated g-fixed point. Then g^* can be written as $\mathrm{diag}(-1, -i)$ near P with appropriate coordinates. Let C be a g-fixed irreducible curve and Q a point in C. Then g^* can be written as $\mathrm{diag}(1, i)$ near Q with appropriate coordinates.

Similarly as in the case $I = 2$ (Lemma 2.12) or $I = 3$ (Lemma 2.10), we can state and prove the following lemma.

Lemma 2.14 ("Four Go" Lemma)**.** *Let (S, g) be a pair of smooth K3 surface S and an automorphism g of S. Assume that $g^4 = \mathrm{id}$ and $g^*\omega_S = i\omega_S$.*

1) *Let $C_1 - C_2 - C_3 - C_4$ be a chain of g-stable smooth rational curves. Then exactly one of C_j is g-fixed, and exactly one of C_k is g^2-fixed but not g-fixed. Moreover, $\{j, k\} = \{1, 3\}$ or $\{2, 4\}$.*

2) *Let C be a g-stable but not h-fixed smooth rational curve on S. Then there exists a unique g-fixed curve D_1 and a unique g^2-fixed but not g-fixed curve D_2 such that $C \cdot D_1 = C \cdot D_2 = 1$.*

3) *Let M and N be the number of smooth rational curves and the number of isolated points in S^g, respectively. Then $M - 2N = 4$.*

Proof. 1) Applying Lemma 2.12 to $h := g^2$, we may assume that C_1, C_3 are h-fixed and C_2, C_4 are not. Note that $\{P\} := C_1 \cap C_2$ and $\{Q\} := C_2 \cap C_3$ are g-fixed. The action of g on the tangent space $T_{C_2,P}$ of C_2 at P is the multiplicative of i or $-i$, and the action of g on $T_{C_2,Q}$ is the multiplicative of $-i$ or i, respectively. For the first case, C_1 is g-fixed and C_3 not; and conversely for the second case.

2) Let P and Q be the g-fixed points on C. Then the actions of g on $T_{C,P}$ and $T_{C,Q}$ are the multiplication of i and $-i$, respectively. So there is a unique g-fixed curve passing through P and a unique h-fixed but not g-fixed curve passing through Q.

3) We can write

$$S^g = \{P_1\} \coprod \cdots \coprod \{P_M\} \coprod C_1 \coprod \cdots \coprod C_N,$$

where P_j are the isolated g-fixed points, and C_k are the smooth irreducible rational g-fixed curves of S. Consider the holomorphic Lefschetz number $L(g)$, which can be evaluated in two different ways.

Method 1. $L(g) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathcal{O}_S)})$ (cf. [2, §3]).

We see that $H^0(S, \mathcal{O}_S) \simeq \mathbb{C}$, $H^1(S, \mathcal{O}_S) = 0$, and by Serre duality

$$H^2(S, \mathcal{O}_S) \simeq H^0(S, \mathcal{O}_S(K_S))^\vee = H^0(S, \mathcal{O}_S)^\vee.$$

Then $g^*|_{H^0(S,\mathcal{O}_S)} = \mathrm{id}$, $g^*|_{H^1(S,\mathcal{O}_S)} = 0$ and $g^*|_{H^2(S,\mathcal{O}_S)} = i^{-1} = -i$.

Method 2. $L(g) = \sum_{j=1}^{M} a(P_j) + \sum_{k=1}^{N} b(C_k)$.

$$a(P_j) := \frac{1}{\det(1 - g^*|_{T_{P_j}})},$$
$$b(C_k) := \frac{1 - \pi(C_k)}{1 - \lambda_k^{-1}} - \frac{\lambda_k^{-1}}{(1 - \lambda_k^{-1})^2}(C_k)^2,$$

where $\pi(C_k)$ is the genus and $(C_k)^2$ is the self-intersection number of C_k, and λ_k is the eigenvalue of g^* on the normal bundle of C_k (cf. [3, §4]).

Recall that $g^*|_{T_{P_j}} = \operatorname{diag}(-1, -i)$. Then

$$a(P_j) = \frac{1}{(1+1)(1+i)} = \frac{1-i}{4}.$$

Since $g^*|_{T_{Q_k}} = \operatorname{diag}(1, i)$ near $Q_k \in C_k$, $\lambda_k = i^{-1}$ is the eigenvalue of g^* on the normal bundle. So

$$b(C_k) = \frac{1-0}{1-i} - \frac{i}{(1-i)^2}(-2) = -\frac{1-i}{2}.$$

Therefore, $1 - i = \dfrac{M}{4}(1-i) - \dfrac{N}{2}(1-i)$; that is, $M - 2N = 4$. □

Now suppose $I(Z) = 4$. Then the associated pair (S, g) satisfies the conditions in Lemmas 2.9 and 2.14. Set $h := g^2$. First of all, we claim that

Lemma 2.15. *With the notations as in Theorem 2 and above, each connected component Δ_i of Δ is h-stable.*

Proof. Suppose Δ_i is not h-stable. Then Δ_i, $g(\Delta_i)$, $h(\Delta_i)$ and $g^3(\Delta_i)$ are distinct components in Δ, and they are contracted to Du Val singular points on $\bar{S}/\langle g\rangle$, a contradiction to our assumption. □

Therefore, applying Proposition 2.8 to (S, h) we have $(S, h) \simeq (S_2, g_2)$, the Shioda-Inose's pair of discriminant 4. From now on, we set $(S, h) = (S_2, g_2)$. Since is known that $(g_2^*)^2 = \mathrm{id}$ on $\mathrm{Pic}(S)$, we can write $g^*|_{\mathrm{Pic}(S)\otimes\mathbb{C}} = \mathrm{diag}(I_s, -I_t)$, where $s + t = \rho(S) = 20$.

Let $x \in T_S$. Suppose $g^*x = \pm x$. Then

$$x \cdot \omega_S = g^*(x \cdot \omega_S) = g^*x \cdot g^*\omega_S = \pm x \cdot i\omega_S = \pm i(x \cdot \omega_S).$$

It follows that $x \cdot \omega_S = 0$. Then $x \in \mathrm{Pic}(S) \cap T_S = \{0\}$. So ± 1 are not eigenvalues of $g^*|_{T_S\otimes\mathbb{C}}$. By Lemma 2.9, we can thus write $g^*|_{T_S\otimes\mathbb{C}} = \mathrm{diag}(i, -i)$.

Proposition 2.16. *With the notations as in Theorem 2. Suppose $I = 4$. Let $h = g^2$. Then $(S, h) \simeq (S_2, g_2)$, the Shioda-Inose's pair of discriminant 4. Moreover, Δ is of the type $A_1 \oplus A_{17}$, $A_5 \oplus A_{13}$ or $A_9 \oplus A_9$.*

Proof. We only need to check the second assertion. Let M be the number of isolated g-fixed points and N the number of smooth irreducible g-fixed curves. By Lemma 2.14, we have $M - 2N = 4$.

Step 1. $N \leq 4$.

We apply the usual topological Lefschetz formula:

$$\chi_{\text{top}}(S^g) = \sum_{i=0}^{4} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathbb{Q})}).$$

The left-hand side is $M + 2N = 4N + 4$, and the right-hand side is

$$2 + \operatorname{tr}(g^*|_{\operatorname{Pic}(S)\otimes\mathbb{C}}) + \operatorname{tr}(g^*|_{T_S\otimes\mathbb{C}}) = 2 + s - t.$$

where $g^*|_{\operatorname{Pic}(S)\otimes\mathbb{C}} = \operatorname{diag}(I_s, -I_t)$. Since $s + t = \rho(S) = 20$, we have

$$s = 11 + 2N \quad \text{and} \quad t = 9 - 2N.$$

It follows that $N \leq 4$.

Step 2. $\Delta = A_{2m-1} \oplus A_{2n-1}$, where $m + n = 10$.

This follows immediately from Proposition 2.13.

Step 3. $\Delta \neq A_3 \oplus A_{15}$ and $\Delta \neq A_7 \oplus A_{11}$. So Proposition 2.16 will follow.

i) Suppose $\Delta = A_3 \oplus A_{15}$. Denote $A_3 = C_1 - C_2 - C_3$ and $A_{15} = D_1 - D_2 - \cdots - D_{15}$. Then it follows from the proof of Proposition 2.13 that all C_i and D_j are h-stable, and from which

$$C_1, C_3, D_1, D_3, D_5, D_7, D_9, D_{11}, D_{13}, D_{15}$$

are h-fixed and others are not.

Clearly each connected component is g-stable, and $\mathrm{Aut}(\Delta) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Note that $g(C_1) = C_1$ or C_3. For each case C_2 is g-stable but not h-fixed. By Lemma 2.14, C_2 intersects with a unique g-fixed curve. Then C_1 or C_3 is g-stable, and therefore all C_i are g-stable. Similarly, by noting that D_8 is g-stable but not h-fixed, we see that all D_j are g-stable. By Lemma 2.14 again, $C_1, D_1, D_5, D_9, D_{13}$ must be g-fixed. But this contradicts $N \leq 4$.

ii) Suppose $\Delta = A_7 \oplus A_{11}$. Denote $A_7 = C_1 - C_2 - \cdots - C_7$ and $A_{11} = D_1 - D_2 - \cdots - D_{11}$. Then using the same argument as for $A_3 \oplus A_{15}$, we can show that C_i and D_j are g-stable for all i, j, and therefore C_1, C_5, D_1, D_5, D_9 are g-fixed. This contradicts $N \leq 4$ again. □

Proof of Theorem 2 (4). It remains to show that $A_1 \oplus A_{17}$, $A_5 \oplus A_{13}$ and $A_9 \oplus A_9$ are realizable.

Let g_4 be the automorphism of S_2 induced by the action $\mathrm{diag}(i, 1)$ on $E^2_{\zeta_4}$. Then $g_4^2 = g_2$ as in Definition 2.6. From the construction of the 24 rational curves in S_2 (Figure 2.2), we see that

I) 4 curves are g_4-fixed, say F_1, F_2 and G_1, G_3;

II) 6 curves are g_2-fixed but not g_4-fixed, say $F_2, G_2, H_{11}, H_{13}, H_{31}, H_{33}$;

III) $g_4(H_{22}) = H'_{22}$ and $g_4(H'_{22}) = H_{22}$;

IV) the remaining 12 curves are g_4-stable, but not g_2-fixed.

Let $g := g_4$ and $h := g^2$. Then Δ contains exactly 4 g-fixed curves (i.e., $N = 4$), and 6 h-fixed but not g-fixed curves. Consider the following three possible types of Δ.

i) $A_1 \oplus A_{17}$.

Since A_1 contains at most 1 g-fixed curve, A_{17} must contain at least 3 g-fixed curves. Then every curve in A_{17} is g-stable. Moreover, it contains 9 h-fixed curves. Noting that Δ has exactly 4 g-fixed curves, we see that C_3, C_7, C_{11}, C_{15} are the g-fixed curves and $C_1, C_5, C_9, C_{13}, C_{17}, A_1$ are the h-fixed but not g-fixed curves.

ii) $A_5 \oplus A_{13}$.

Since A_5 contains at most 2 g-fixed curves, A_{13} has a g-fixed curve. So every curve in A_{13} is g-stable. We write

$$A_5 = C_1 - C_2 - C_3 - C_4 - C_5,$$

$$A_{13} = D_1 - D_2 - D_3 - \cdots - D_{13}.$$

If C_1 is not g-stable, then only C_3 in A_5 is h-fixed. Note that it is not g-fixed. Then A_{13} shall contain 4 g-fixed curves: D_1, D_5, D_9, D_{13}. However, Δ would have only 5 h-fixed but not g-fixed curves $D_3, D_7, D_{11}, D_{15}, C_3$, which is a contradiction.

Therefore, every curve in A_5 is g-stable. Then A_5 contains at least 1 g-fixed curve, and A_{13} contains at most 3 g-fixed curves. It follows that exactly 4 curves C_3, D_3, D_7, D_{11} in Δ are g-fixed.

iii) $A_9 \oplus A_9$.

We call the second A_9 as A_9'. If A_9 is not g-stable, then $g(A_9) = A_9'$ and $g(A_9') = A_9$. There would be no g-fixed curve in Δ, absurd. So both A_9 and A_9' are g-stable.

Since A_9 contains at most 3 g-fixed curves, A_9' contains at least 1 g-fixed curve. Hence every curve in A_9' is g-stable. Similarly, every curve in A_9 is g-stable. On the other hand, A_9 should contain at least 2 g-fixed curves, so does A_9'. If we write

$$A_9 = C_1 - C_2 - C_3 - \cdots - C_9,$$

$$A_9' = D_1 - D_2 - D_3 - \cdots - D_9,$$

then exactly C_3, C_7, D_3 and D_7 are g-fixed.

Since we have determined the action of g on Δ and these Δ can be obtained from the 22 g-stable rational curves in S_2 (Figure 2.2), they are all realizable. The dual graphs are given in Table 2.2 (1), (3) and (5). $\square$

Note that in the proof of above, we showed that for each of the every cases, every irreducible curve in Δ is g-stable.

2.4.4 Impossibility of $I = 6$

In order to complete the proof of Theorem 2, in this section we will explore the method used in [33, Proposition 2.12, Lemma 2.13] to prove the following.

Proposition 2.17. *With the notations in Theorem 2, $I \neq 6$.*

Proof. We assume that there is a log Enriques surface Z of rank 18 without Du Val singularities. Let (S, g) be the associated pair. Let P be an isolated g-fixed point. Then g^* can be written as either

i) $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$, or

ii) $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$

with appropriate coordinates around P.

Step 1. There are even number of isolated g-fixed points of the second type.

Suppose $g^* = \mathrm{diag}(\zeta_6^2, \zeta_6^5)$ near P. Then $(g^2)^* = (\zeta_6^4, \zeta_6^4)$ near P. It follows that P is an isolated g^2-fixed point.

Suppose $g^* = \mathrm{diag}(\zeta_6^3, \zeta_6^4)$ near P. Then $(g^2)^* = \mathrm{diag}(1, \zeta_6^2)$, and there exists a unique smooth rational g^2-fixed curve C passing through P. Since S^{g^2} is smooth, C is g-stable but not g-fixed. Let Q be the other g-fixed point on C. Since Q is not an isolated g^2-fixed point, it is also an isolated g-fixed point of the second type.

Therefore, the g-fixed points of the second type come in pairs. There are even number of such points.

Step 2. The number of isolated g-fixed points of the first type equals that of the second type.

Let P be an isolated g-fixed point. Since $S^g \subseteq S^{g^3}$, a disjoint union of smooth rational curves, there is a unique g^3-fixed curve C passing through P. Hence, C is g-stable but not g-fixed, and it contains exactly 2 g-fixed points. Note that if P is of the first type $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$, then $g^*|_{T_{C,P}} = \zeta_6^2$; if P is of the second type $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$, then $g^*|_{T_{C,P}} = \zeta_6^4$. So the other isolated g-fixed point on C is of different type of P.

Therefore, there is a one-to-one correspondence between the set of g-fixed points of the first type and that of the second type. Step 2 is proved.

Now we can set $P_1, \ldots, P_{2\ell}$ and $Q_1, \ldots, Q_{2\ell}$ to be the isolated S^g-fixed points of type $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$ and of type $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$, respectively.

Suppose there are c rational smooth g-fixed curves, say $C_1, \ldots, C_c$. We claim that

Step 3. $\ell = c + 1$.

Similarly as in the proof of Lemma 2.14, we use the holomorphic Lefschetz fixed point formula

$$L(g) = \sum_{i=0}^{2} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathcal{O}_S)}) = \sum_{i=1}^{2\ell} a(P_i) + \sum_{i=1}^{2\ell} a(Q_i) + \sum_{i=1}^{c} b(C_i).$$

We can compute that

$$\sum_{i=0}^{2} (-1)^i \operatorname{tr}(g^*|_{H^i(S,\mathcal{O}_S)}) = 1 + 0 + \frac{1}{\zeta_6} = \frac{3 - i\sqrt{3}}{2}.$$

$$\begin{aligned}
a(P_i) &= \frac{1}{\det(1 - g^*|_{T_{P_i}})} = \frac{1}{(1-\zeta_6^2)(1-\zeta_6^5)} = \frac{3 - i\sqrt{3}}{6}, \\
a(Q_i) &= \frac{1}{\det(1 - g^*|_{T_{Q_i}})} = \frac{1}{(1-\zeta_6^3)(1-\zeta_6^4)} = \frac{3 - i\sqrt{3}}{12}, \\
b(C_i) &= \frac{1 - \pi(C_i)}{1 - \zeta_6} - \frac{\zeta_6\, C_i^2}{(1-\zeta_6)^2} = -\frac{3 - i\sqrt{3}}{2}.
\end{aligned}$$

Therefore, $\ell = c + 1$.

Step 4. Determine S^{g^2}.

If P is a g^2-fixed but not g-fixed point, then so is $g(P)$. Therefore, there are even number of g^2-fixed but not g-fixed points. If C is a rational smooth irreducible g^2-fixed curve which does not contain any g-fixed point, so is $g(C)$. Hence, there are even number of such curves.

Suppose the isolated g^2-fixed points are $P_1, \dots, P_{2c+2}, R_1, \dots, R_{2k}$, and the smooth rational g^2-fixed curves are $C_1, \dots, C_c, D_1, \dots, D_{c+1}, \dots, F_1, \dots, F_{2p}$, where R_i is not g-fixed, $Q_{2i-1}, Q_{2i} \in D_i$, and F_i does not contain at g-fixed point.

Then apply Lemma 2.10 to (S, g^2), we obtain

$$(2c + 2 + 2k) - (c + c + 1 + 2p) = 3,$$

which implies $k = p + 1$.

Step 5. Determine S^{g^3}.

We note g^3 is a non-symplectic involution on S, and so there is no isolated g^3-fixed point. If G is a g^3-fixed curve which does not contain any g-fixed point, then so are $g(G)$ and $g^2(G)$. Therefore, the smooth rational g^3-fixed curves are $C_1, \dots, C_c$, $E_1, \dots, E_{2c+2}$, $G_1, \dots, G_{3q}$, where $P_i, Q_i \in E_i$ and G_i does not contain any g-fixed point.

Step 6. $c + p + q \leq 2$.

Since $\mathrm{ord}(g) = 6$, we can write

$$g^*|_{H^2(S,\mathbb{Q})} = \mathrm{diag}(I_\alpha, -I_\beta, \zeta_6^2 I_\gamma, \bar{\zeta}_6^2 I_\gamma, \zeta_6 I_{1+\delta}, \bar{\zeta}_6 I_{1+\delta}),$$

where $\alpha, \beta, \gamma, \delta \geq 0$.

Let $j = 1$ in the topological Lefschetz fixed point formula

$$\chi_{\text{top}}(S^{g^j}) = \sum_{i=0}^{4} (-1)^i \operatorname{tr}((g^j)^*|_{H^i(S,\mathbb{Q})}).$$

We have

$$(2c + 2) + (2c + 2) + 2 \cdot c = 2 + \alpha - \beta - \gamma + (\delta + 1).$$

$(g^2)^*|_{H^2(S,\mathbb{Q})} = \operatorname{diag}(I_{\alpha+\beta}, \zeta_6^2 I_{\gamma+\delta+1}, \bar{\zeta}_6^2 I_{\gamma+\delta+1})$. Then for $j = 2$ we have

$$(2c + 2) + (2p + 2) + 2[c + (c + 1) + 2p] = 2 + (\alpha + \beta) - (\gamma + \delta + 1).$$

$(g^3)^*|_{H^2(S,\mathbb{Q})} = \operatorname{diag}(I_{\alpha+2\gamma}, -I_{\beta+2+2\delta})$. Then for $j = 3$ we have

$$2[c + (2c + 2) + 3q] = 2 + (\alpha + 2\gamma) - (\beta + 2 + 2\delta).$$

We also note that

$$\alpha + \beta + 2\gamma + 2(1 + \delta) = \dim H^2(S, \mathbb{Q}) = 22.$$

It can be solved that $\delta = -c - p - q + 2$. In particular, $c + p + q \leq 2$.

Step 7. Determine the possible types of Δ.

Let Δ_i be a connected component of Δ. Then Δ_i is either g^3-stable or g^2-stable, otherwise $g^k(\Delta_i)$, $k = 0, \ldots, 5$, would be contracted to a single Du Val singular point in $\bar{S}/\langle g \rangle$, which should not exist by assumption.

Suppose Δ_i, $i = 1, \ldots, m$, are the g^3-stable connected components of Δ. Since $(g^3)^*\omega_S = -\omega_S$, using the same argument as for $I = 2$, we see that $\Delta_i = A_{2m_i-1}$ for some m_i, which contains exactly m_i smooth rational g^3-fixed curves. On the other hand, by computation in Step 4, there are $c + (2c + c) + 3q = 3(c + q) + 2$ g-fixed curves. Therefore,

$$\sum_{i=1}^{n} \operatorname{rank} \Delta_i = \sum_{i=1}^{m} (2m_i - 1) = 6(c + q) + 4 - m.$$

Since $\ell = c + 1 > 0$, $S^g \neq \emptyset$. We see that $m \geq 1$.

Suppose Δ'_j, $j = 1, \ldots, n$, are the g^2-stable but not g-stable connected components of Δ. Since $(g^2)^* \omega_S = \zeta_3 \omega_S$, using the same argument as for $I = 3$, we see that each Δ'_j has Dynkin type A or D.

Since each Δ'_j contains at least one g^2-fixed curve and $F_1, \ldots, F_{2p}$ are the only g^2-fixed curves in Δ'_j, we have $n \leq 2p$. On the other hand, from the proof of Proposition 2.11 Step 4, if $\operatorname{rank} \Delta'_j = \alpha_j$, then Δ_j contains at least $\lceil (\alpha_j - 1)/3 \rceil$ smooth g^2-fixed curves.

We have an estimation

$$2p \geq \sum_{j=1}^{n} \lceil (\alpha_j - 1)/3 \rceil \geq \sum_{j=1}^{n} (\alpha_j - 1)/3.$$

That is,

$$\sum_{j=1}^{n} \operatorname{rank} \Delta'_j \leq 6p + n.$$

Note that Δ'_j is not g^3-stable, otherwise it would also be g-stable. So Δ'_j and $g^3(\Delta'_j)$ are disjoint connected components in Δ. In particular, n is even.

It follows from $\operatorname{rank}\Delta = 18$ that

$$\begin{aligned}18 &\le 6(c+q)+4-m+6p+n = 6(c+p+q)+4-m+n\\ &\le 6\cdot 2+4-m+n = 16-m+n\\ &\le 16-1+n = 15+n\\ &\le 15+2p.\end{aligned}$$

Then $p \ge 2$ and it follows from $c+p+q \le 2$ that $p = 2$ and $c = q = 0$. So Δ has no g-fixed curve. Since n is even, $n = 4$ and $m = 1$ or 2.

We are left to show that these two cases are impossible.

Recall that Δ_i has the form A_{2m_i-1} and contains exactly m_i g^3-fixed curves, and the 2 irreducible g^3-fixed curves are contained in $\coprod_{i=1}^m \Delta_i$. We have $\sum_{i=1}^m m_i = 2$.

If $m = 1$, then $m_1 = 2$ and $\Delta_1 = A_3$. However, this would imply that $\sum_{j=1}^4 \operatorname{rank}\Delta'_j = 15$, which needs to be even.

If $m = 2$, then $m_1 = m_2 = 1$ and $\Delta_1 = \Delta_2 = A_1$. They are g^3-fixed. On the other hand, note that $\operatorname{ord}(g^2) = 3$. By considering the g^2-action on Δ, we see that Δ_1 and Δ_2 are also g^2-fixed. It follows that Δ_1 and Δ_2 g-fixed, which contradicts our computation that there is no g-fixed curve. $\square$

This completes the proof of Proposition 2.17 and also Theorem 2 (1).

2.5 The List of Dynkin's Types of Δ

Table 2.1: $I = 3$

"f" denotes the g-fixed curve and s denotes the g-stable but not g-fixed curve. We use the same labeling for curves as in Figure 2.1.

(A) Realizable Cases.

Case I: A_{18}: $s-f-s-s-f-s-s-f-s-s-f-s-s-f-s-s-f-s$

$E_{33}-G_3-E_{13}-E'_{13}-F_1-E'_{11}-E_{11}-G_1-E_{31}-E'_{31}-F_3-E'_{32}-E_{32}-G_2-E_{22}-E'_{22}-F_2-E'_{21}$.

Case II: D_{18}: $\begin{matrix} s \\ s \end{matrix} > f-s-s-f-s-s-f-s-s-f-s-s-f-s-s-f$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1-E'_{13}-E_{13}-G_3-E_{33}-E'_{33}-F_3-E'_{31}-E_{31}-G_1-E_{21}-E'_{21}-F_2-E'_{22}-E_{22}-G_2$

Case III: $A_{3m} \oplus A_{3n}$, where $m+n=6$, $1 \le m \le n \le 5$.

(1) $A_3 \oplus A_{15}$: $s-f-s$, $s-f-s-s-f-s-s-f-s-s-f-s-s-f-s$

$E'_{11}-F_1-E'_{12}$

$E_{13}-G_3-E_{33}-E'_{33}-F_3-E'_{31}-E_{31}-G_1-E_{21}-E'_{21}-F_2-E'_{22}-E_{22}-G_2-E_{32}$

(2) $A_6 \oplus A_{12}$: $s-f-s-s-f-s$, $s-f-s-s-f-s-s-f-s-s-f-s$

$E_{21}-G_1-E_{11}-E'_{11}-F_1-E'_{12}$

$E_{13}-G_3-E_{23}-E'_{23}-F_2-E'_{22}-E_{22}-G_2-E_{32}-E'_{32}-F_3-E'_{33}$

(3) $A_9 \oplus A_9$: $s-f-s-s-f-s-s-f-s$, $s-f-s-s-f-s-s-f-s$

$E'_{11}-F_1-E'_{12}-E_{12}-G_2-E_{22}-E'_{22}-F_2-E'_{23}$

$E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

Case IV: $D_{3m} \oplus A_{3n}$, where $m + n = 6$.

(1) $D_6 \oplus A_{12}$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $s - f - s - s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

$E'_{33} - F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

(2) $D_9 \oplus A_9$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$, $s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2$

$E_{22} - G_2 - E_{32} - E'_{32} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

(3) $D_{12} \oplus A_6$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$, $s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{22} - E_{22} - G_2$

$E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

(4) $D_{15} \oplus A_3$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s - s - f$, $s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3$

$E_{22} - G_2 - E_{32}$

Case V: $D_{3m} \oplus D_{3n}$, where $m + n = 6$, $2 \leq m \leq n \leq 4$.

(1) $D_6 \oplus D_{12}$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$.

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

$\begin{matrix} E'_{33} \\ E'_{32} \end{matrix} > F_3 - E'_{31} - E_{31} - G_1 - E_{21} - E'_{21} - F_2 - E'_{22} - E_{22} - G_2$

Case VI: $D_{3n+1} \oplus A_{3m-1}$, $m + n = 6$, $1 \leq m, n \leq 5$.

(1) $D_4 \oplus A_{14}$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $f - s - s - f - s - s - f - s - s - f - s - s - f - s$

$$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$$

$$G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22}$$

(2) $D_7 \oplus A_{11}$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $f - s - s - f - s - s - f - s - s - f - s$

$$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23}$$

$$G_2 - E_{22} - E'_{22} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33}$$

(3) $D_{10} \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s$, $f - s - s - f - s - s - f - s$

$$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21}$$

$$G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22}$$

(4) $D_{13} \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, $f - s - s - f - s$

$$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$$

$$G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$$

(5) $D_{16} \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, $f - s$

$$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$$

Case VII: $A_{3m} \oplus A_{3n} \oplus A_{3r}$, $m + n + r = 6$, $1 \le m \le n \le r \le 4$.

(1) $A_3 \oplus A_3 \oplus A_{12}$: $s - f - s$, $s - f - s$, $s - f - s - s - f - s - s - f - s - s - f - s$

$$E_{13} - G_3 - E_{23}$$

$$E'_{32} - F_3 - E'_{33}$$

$$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22} - E'_{22} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$$

(2) $A_3 \oplus A_6 \oplus A_9$: $s - f - s$, $s - f - s - s - f - s$, $s - f - s - s - f - s - s - f - s$

$$E_{13} - G_3 - E_{33}$$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$

$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$

(3) $A_6 \oplus A_6 \oplus A_6$: $s - f - s - s - f - s$, $s - f - s - s - f - s$, $s - f - s - s - f - s$

$E'_{11} - F_1 - E'_{12} - E_{12} - G_2 - E_{22}$

$E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{32}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

Case VIII: $D_6 \oplus D_6 \oplus D_6$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

$\begin{matrix} E'_{21} \\ E'_{23} \end{matrix} > F_2 - E'_{22} - E_{22} - G_2$

$\begin{matrix} E'_{32} \\ E'_{33} \end{matrix} > F_3 - E'_{31} - E_{31} - G_1$

Case X: $A_{3m} \oplus A_{3n} \oplus D_{3r}$, where $m + n + r = 6$, $m \leq n$.

(1) $A_3 \oplus A_3 \oplus D_{12}$: $s - f - s$, $s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$

$E_{22} - G_2 - E_{32}$

$E'_{31} - F_3 - E'_{33}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2 - E'_{21} - E_{21} - G_1$

(2) $A_3 \oplus A_6 \oplus D_9$: $s - f - s$, $s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$

$E_{22} - G_2 - E_{32}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{23} - E'_{23} - F_2$

(3) $A_3 \oplus A_9 \oplus D_6$: $s - f - s$, $s - f - s - s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$.

$E_{22} - G_2 - E_{32}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

(4) $A_6 \oplus A_6 \oplus D_6$: $s - f - s - s - f - s$, $s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - f - s$

$E_{22} - G_2 - E_{32} - E'_{32} - F_3 - E'_{33}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31}$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3$

Case XI: $D_{3m+1} \oplus A_{3n} \oplus A_{3r-1}$, where $m + n + r = 6$.

(1) $D_4 \oplus A_3 \oplus A_{11}$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s$, $f - s - s - f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31}$

$F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

(2) $D_4 \oplus A_6 \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s - s - f - s$, $f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$

$G_2 - E_{22} - E'_{22} - F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

(3) $D_4 \oplus A_9 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $s - f - s - s - f - s - s - f - s$, $f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{33} - E_{33} - G_3 - E_{23}$

$F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

(4) $D_4 \oplus A_{12} \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s,\ s - f - s - s - f - s - s - f - s - s - f - s,\ f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2 - E_{22} - E'_{22} - F_2 - E'_{23}$

$G_3 - E_{33}$

(5) $D_7 \oplus A_3 \oplus A_8$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ s - f - s,\ f - s - s - f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$E_{22} - G_2 - E_{32}$

$F_3 - E'_{31} - E_{31} - G_1 - E_{21} - E'_{21} - F_2 - E'_{23}$

(6) $D_7 \oplus A_6 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ s - f - s - s - f - s,\ f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$E'_{23} - F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

$F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

(7) $D_7 \oplus A_9 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s,\ s - f - s - s - f - s - s - f - s,\ f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$E'_{23} - F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3 - E'_{32}$

$G_2 - E_{22}$

(8) $D_{10} \oplus A_3 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s,\ s - f - s,\ f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31}$

$E_{22} - G_2 - E_{32}$

$G_1 - E_{21} - E'_{21} - F_2 - E'_{23}$

(9) $D_{10} \oplus A_6 \oplus A_2$: $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s - s - f - s,\ s - f - s - s - f - s,\ f - s$

$\begin{smallmatrix} E'_{11} \\ E'_{12} \end{smallmatrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31}$

$E'_{23} - F_2 - E'_{22} - E_{22} - G_2 - E_{32}$

$G_1 - E_{21}$

(10) $D_{13} \oplus A_3 \oplus A_2$: $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s - s - f - s,\ s - f - s,\ f - s$

$\begin{smallmatrix} E'_{11} \\ E'_{12} \end{smallmatrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1 - E_{21}$

$E_{22} - G_2 - E_{32}$

$F_2 - E'_{23}$

Case XII: $D_{3m+1} \oplus D_{3n+1} \oplus A_{3r-2}$, where $m + n + r = 6$, $m \leq n$.

(2) $D_4 \oplus D_7 \oplus A_7$: $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s,\ \begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s,\ f - s - s - f - s - s - f$

$\begin{smallmatrix} E'_{11} \\ E'_{12} \end{smallmatrix} > F_1 - E'_{13}$

$\begin{smallmatrix} E'_{21} \\ E'_{22} \end{smallmatrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

$G_1 - E_{31} - E'_{31} - F_3 - E'_{32} - E_{32} - G_2$

(5) $D_7 \oplus D_7 \oplus A_4$: $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s,\ \begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s,\ f - s - s - f$

$\begin{smallmatrix} E'_{12} \\ E'_{13} \end{smallmatrix} > F_1 - E'_{11} - E_{11} - G_1 - E_{31}$

$\begin{smallmatrix} E'_{21} \\ E'_{22} \end{smallmatrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33}$

$G_2 - E_{32} - E'_{32} - F_3$

(6) $D_7 \oplus D_{10} \oplus A_1$: $\begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s,\ \begin{smallmatrix} s \\ s \end{smallmatrix} > f - s - s - f - s - s - f - s,\ f$

$\begin{smallmatrix} E'_{12} \\ E'_{13} \end{smallmatrix} > F_1 - E'_{11} - E_{11} - G_1 - E_{31}$

$\begin{smallmatrix} E'_{21} \\ E'_{22} \end{smallmatrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33} - E'_{33} - F_3 - E'_{32}$

G_2

Case XIII: $D_{3n+1} \oplus D_{3m} \oplus A_{3r-1}$, where $m + n + r = 6$, $m \geq 2$.

(3) $D_4 \oplus D_{12} \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13}$

$\begin{matrix} E'_{21} \\ E'_{22} \end{matrix} > F_2 - E'_{23} - E_{23} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31} - E_{31} - G_1$

$G_2 - E_{32}$

(4) $D_7 \oplus D_6 \oplus A_5$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $f - s - s - f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$\begin{matrix} E'_{22} \\ E'_{23} \end{matrix} > F_2 - E'_{21} - E_{21} - G_1$

$G_2 - E_{32} - E'_{32} - F_3 - E'_{31}$

(5) $D_7 \oplus D_9 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33}$

$\begin{matrix} E'_{22} \\ E'_{23} \end{matrix} > F_2 - E'_{21} - E_{21} - G_1 - E_{31} - E'_{31} - F_3$

$G_2 - E_{32}$

(6) $D_{10} \oplus D_6 \oplus A_2$: $\begin{matrix} s \\ s \end{matrix} > f - s - s - f - s - s - f - s - s - f - s$, $\begin{matrix} s \\ s \end{matrix} > f - s - s - f$, $f - s$

$\begin{matrix} E'_{11} \\ E'_{12} \end{matrix} > F_1 - E'_{13} - E_{13} - G_3 - E_{33} - E'_{33} - F_3 - E'_{31}$

$\begin{matrix} E'_{22} \\ E'_{23} \end{matrix} > F_2 - E'_{21} - E_{21} - G_1$

$G_2 - E_{32}$

(B) Indeterminate Cases

Case V: (2) $D_9 \oplus D_9$: $\substack{s\\ s}\!> f - s - s - f - s - s - f,\ \substack{s\\ s}\!> f - s - s - f - s - s - f$

Case IX: (1) $A_3 \oplus D_6 \oplus D_9$: $s - f - s,\ \substack{s\\ s}\!> f - s - s - f,\ \substack{s\\ s}\!> f - s - s - f - s - s - f$

Case IX: (2) $A_6 \oplus D_6 \oplus D_6$: $s - f - s - s - f - s,\ \substack{s\\ s}\!> f - s - s - f,\ \substack{s\\ s}\!> f - s - s - f$

Case XII: (1) $D_4 \oplus D_4 \oplus A_{10}$: $\substack{s\\ s}\!> f - s,\ \substack{s\\ s}\!> f - s,\ f - s - s - f - s - s - f - s - s - f$

Case XII: (3) $D_4 \oplus D_{10} \oplus A_4$: $\substack{s\\ s}\!> f - s,\ \substack{s\\ s}\!> f - s - s - f - s - s - f - s,\ f - s - s - f$

Case XII: (4) $D_4 \oplus D_{13} \oplus A_1$: $\substack{s\\ s}\!> f - s,\ \substack{s\\ s}\!> f - s - s - f - s - s - f - s - s - f - s,\ f$

Case XIII: (1) $D_4 \oplus D_6 \oplus A_8$: $\substack{s\\ s}\!> f - s,\ \substack{s\\ s}\!> f - s - s - f,\ f - s - s - f - s - s - f - s$

Case XIII: (2) $D_4 \oplus D_9 \oplus A_5$: $\substack{s\\ s}\!> f - s,\ \substack{s\\ s}\!> f - s - s - f - s - s - f,\ f - s - s - f - s.$

Table 2.2: $I = 2, 4$

We use the same labeling as in Figure 2.2. For $I = 2$, "f" denotes the g-fixed curve and s denotes the g-stable but not g-fixed curve. For $I = 4$, define $h = g^2$; "f" denotes the g-fixed curve, "h" denotes the h-fixed but not g-fixed curve and "s" denotes the g-stable but not h-fixed curve.

(1) $A_1 \oplus A_{17}$:

$I = 2$: $f,\ f - s - f - s - f - s - f - s - f - s - f - s - f - s - f - s - f$

$I = 4$: $h,\ h - s - f - s - h - s - f - s - h - s - f - s - h - s - f - s - h$

H_{11}

$H_{13} - E'_{13} - F_1 - E_{12} - G_2 - E_{32} - F_3 - E'_{33} - H_{33} - G_3 - E'_{23} - F_2 - E'_{21} - G_1 - E_{31} - H_{31}.$

(2) $A_3 \oplus A_{15}$:

$I = 2$: $f - s - f,\ f - s - f - s - f - s - f - s - f - s - f - s - f - s - f$

$F_2 - E_{22} - G_2$

$H_{11} - E'_{11} - F_1 - E'_{13} - H_{13} - E_{13} - G_3 - E_{33} - H_{33} - E'_{33} - F_3 - E'_{31} - H_{31} - E_{31} - G_1.$

(3) $A_5 \oplus A_{13}$:

$I = 2$: $f - s - f - s - f$, $f - s - f - s - f - s - f - s - f - s - f - s - f$

$I = 4$: $h - s - f - s - h$, $h - s - f - s - h - s - f - s - h - s - f - s - h$

$H_{13} - E_{13} - G_3 - E_{33} - H_{33}$

$H_{11} - E'_{11} - F_1 - E_{12} - G_2 - E_{32} - F_3 - E'_{31} - H_{31} - E_{31} - G_1 - E'_2 - F_2$

(4) $A_7 \oplus A_{11}$:

$I = 2$: $f - s - f - s - f - s - f$, $f - s - f - s - f - s - f - s - f - s - f$

$H_{13} - E_{13} - G_3 - E_{33} - H_{33} - E'_{33} - F_3$

$H_{11} - E'_{11} - F_1 - E_{12} - G_2 - E_{22} - F_2 - E'_{21} - G_1 - E_{31} - H_{31}$

(5) $A_9 \oplus A_9$:

$I = 2$: $f - s - f - s - f - s - f - s - f$, $f - s - f - s - f - s - f - s - f$

$I = 4$: $h - s - f - s - h - s - f - s - h$, $h - s - f - s - h - s - f - s - h)$

$H_{11} - E'_{11} - F_1 - E_{12} - G_2 - E_{32} - F_3 - E'_{33} - H_{33}$

$H_{13} - E_{13} - G_3 - E'_{23} - F_2 - E'_{21} - G_1 - E_{31} - H_{31}$

Chapter 3

Dynamics of Automorphism Groups of Projective Varieties

3.1 Introduction

We work over the field of complex numbers $\mathbb{C}$. Let X be a compact manifold. Then $\mathrm{Aut}(X)$, the automorphism group of X, has a natural structure of a finite dimensional Lie group acting holomorphically on X. We use $\mathrm{Aut}_0(X)$ for the *identity component* of $\mathrm{Aut}(X)$, i.e., the irreducible component of $\mathrm{Aut}(X)$ containing the identity $\mathrm{id}_X \in \mathrm{Aut}(X)$. In particular, if X is projective, then $\mathrm{Aut}_0(X)$ has a natural structure of an algebraic group acting algebraically on X.

In this chapter, we are going to prove some results on the relation between the geometry of a compact Kähler manifold or a projective manifold and its automorphism group.

With the Fubini study metric, a projective manifold is necessarily a compact Kähler manifold. Theorems 3.1–3.2 can be applied to both projective manifolds and compact Kähler manifolds, while for the latter case, we consider $H^2(X, \mathbb{Z})$ instead of the Néron-Severi group $\mathrm{NS}(X)$. Theorem 3.3–3.6 are specialized on projective manifolds of dimension 2 or 3.

The analogue of Tits alternative was first noticed and studied by Oguiso [28] for compact Hyperkähler manifolds. Theorem 3.1 extends the results of [44, Theorem 1.1], a theorem of Tits type for compact Kähler manifolds. In particular, Theorem 3.1 implies the Tits alternative for automorphism groups of compact Kähler manifolds, while the classical Tits alternative [35, Theorem 1] treats linear groups.

We now state the main results of this chapter. We refer to Section 3.2 for the notations and terminologies. The proofs of some related results will be given in Section 3.3. In the following, $G \le H$ means G is a subgroup of H.

Theorem 3.1 proved a conjecture of the Tits type ([17, Conjecture 1.3]).

Theorem 3.1. *Let X be a compact Kähler (resp. projective) manifold of dimension n, and $G \le \mathrm{Aut}(X)$ a subgroup of automorphisms. Let $L_{\mathbb{C}} = H^2(X, \mathbb{C})$ (resp. $L_{\mathbb{C}} = \mathrm{NS}_{\mathbb{C}}(X)$). Then exactly one of the following assertions holds:*

1) *$G|_{L_{\mathbb{C}}} \ge \mathbb{Z} * \mathbb{Z}$, and hence $G \ge \mathbb{Z} * \mathbb{Z}$.*

2) *$G|_{L_{\mathbb{C}}}$ is virtually solvable and $G \ge K \cap L(\mathrm{Aut}_0(X)) \ge \mathbb{Z} * \mathbb{Z}$, where $L(\mathrm{Aut}_0(X))$ is the linear part of $\mathrm{Aut}_0(X)$ and $K := \mathrm{Ker}(G \to \mathrm{GL}(L_{\mathbb{C}}))$, so X is ruled.*

3) *There is a finite-index solvable subgroup G_1 of G such that the null subset $N(G_1)$*

of G_1 is a normal subgroup of G_1 and that $G_1/N(G_1) \cong \mathbb{Z}^{\oplus r}$ for some $r \le n-1$.

In particular, either $G \ge \mathbb{Z} * \mathbb{Z}$ or G is virtually solvable. In Cases (2) and (3) above, $G|_{L_\mathbb{C}}$ is finitely generated.

Theorem 3.2 below gives criteria for $G|_{L_\mathbb{C}}$ to be virtually solvable (cf. Definition 3.4). Indeed, (1) is shown in the proof of [44, Theorem 1.2]. We state it here for comparison purpose, and will prove (2) and (3) in Section 3.3.2.

Theorem 3.2. *Let X be a compact Kähler (resp. projective) manifold of dimension n, and $G \le \mathrm{Aut}(X)$ a subgroup. Let $L_\mathbb{C} = H^2(X, \mathbb{C})$ (resp. $L_\mathbb{C} = \mathrm{NS}_\mathbb{C}(X)$. We have the following assertions.*

1) *Suppose that $G|_{L_\mathbb{C}}$ is virtually solvable and its Zariski-closure in $\mathrm{GL}(L_\mathbb{C})$ is connected. Then G is polarized by a quasi-nef sequence $L_1 \cdots L_k$, $(1 \le k \le n-1)$.*

2) *Conversely, suppose that G is polarized by a quasi-nef sequence $L_1 \cdots L_k$ $(1 \le k \le n-1)$. Then $G|_{L_\mathbb{C}}$ is virtually solvable.*

3) *Suppose that $G|_{L_\mathbb{C}}$ is virtually solvable. Then there is a finite-index subgroup G_1 of G such that $N(G_1) \lhd G$ and $G_1/N(G_1) \cong \mathbb{Z}^{\oplus r}$ for some $r \le n-1$.*

Theorem 3.3 is a finer version of Tits alternative theorem by considering the case when X is a smooth projective surface.

Theorem 3.3. *Let X be a smooth projective surface and $G \le \mathrm{Aut}(X)$. Then*

1) *$G|_{\mathrm{NS}_\mathbb{C}(X)} \ge \mathbb{Z} * \mathbb{Z}$ (and hence $G \ge \mathbb{Z} * \mathbb{Z}$, and $G \ne N(G)$), or*

2) $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ *is almost abelian of rank* $r \le \max\{1, \operatorname{rank}\mathrm{NS}_{\mathbb{Q}}(X) - 2\}$. (*Thus, G itself is almost abelian of the same rank r when* $\mathrm{Aut}_0(X) = 1$).

For a surface X, Theorem 3.4 gives a clear geometric interpretation for a group $G \le \mathrm{Aut}(X)$ of null entropy. Note that Theorem 3.4 cannot be generalized to higher dimension by considering $X = X_1 \times X_2$ and $G = G_1 \times G_2$, where $G_1 \le \mathrm{Aut}(X_1)$ is of null entropy while $G_2 \le \mathrm{Aut}(X_2)$ is not.

Theorem 3.4. *Let X be a smooth projective surface and $G \le \mathrm{Aut}(X)$ a subgroup such that $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is an infinite group. Then G is of null entropy if and only if there is a G-equivariant fibration $X \to B$ onto a nonsingular projective curve.*

Theorem 3.5 is in the spirit of [38, Theorem 1.2.1]. We note that $g_1^t = g_2^t$ implies that $g_1^t g_2^t = g_2^t g_1^t$, while the latter implies $\mathrm{Prep}(g_1) = \mathrm{Prep}(g_2)$ (cf. Definition 3.6), which is known to be Zariski-dense since g_i is of positive entropy (cf. [7, §3]).

Theorem 3.5. *Let X be a smooth projective surface and $g_i \in \mathrm{Aut}(X)$ $(i = 1, 2)$ of positive entropy. Suppose that both g_i are polarized by the same nonzero nef $\mathbb{R}$-divisor L. Then we have*

1) $g_1^{s_1} = g_2^{s_2}$ *holds in* $\mathrm{Aut}(X)|_{\mathrm{NS}_{\mathbb{C}}(X)}$ *for some* $s_i \in \mathbb{Z}\backslash\{0\}$.

2) *Suppose that either* $\mathrm{Prep}(g_1) \cap \mathrm{Prep}(g_2) \ne \emptyset$, *or X is not birational to an abelian surface. Then* $g_1^{t_1} = g_2^{t_2}$ *holds in* $\mathrm{Aut}(X)$ *for some* $t_i \in \mathbb{Z}\backslash\{0\}$.

Theorem 3.6 shows that a Kähler manifold has lots of symmetries only when it is a torus or its quotient. Compare it with [46, Theorem 1.1, 1.5] where one has assumed instead that there are $\dim X - 1$ independent and commuting automorphisms of positive entropy.

Theorem 3.6. *Let X be a projective manifold of dimension 3, and $G \leq \mathrm{Aut}(X)$ a subgroup such that $G_0 := G \cap \mathrm{Aut}_0(X)$ is infinite and the quotient group G/G_0 is an almost abelian group of rank $r > 0$. Suppose that the pair (X, G) is strongly primitive. Then X is a complex torus and G_0 is Zariski-dense in $\mathrm{Aut}_0(X)$ ($\cong X$).*

3.2 Preliminaries

In this section we state the definitions used in Chapter 3. Let X be a compact Kähler (resp. projective) manifold, and $\mathrm{Aut}(X)$ the group of automorphisms.

For $g \in \mathrm{Aut}(X)$, its *spectral radius* is defined by

$$\rho(g) := \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } g^*|_{\bigoplus_{i\geq 0} H^i(X,\mathbb{C})}\right\}.$$

It is known that either $\rho(g) > 1$, or $\rho(g) = 1$ and all the eigenvalues of $g^*|_{\bigoplus_{i\geq 0} H^i(X,\mathbb{C})}$ lie on the unit circle. By the fundamental work of Yomdin [37], Gromov [11] and Friedland [8], the topological entropy of an automorphism on a compact Kähler manifold can be defined in several equivalent ways: topological, differential-geometrical, and cohomological. In this chapter, we will use the cohomological definition:

Definition 3.1. Let $g \in \mathrm{Aut}(X)$. Then the (*topological*) *entropy* of g is defined by $h(g) := \log \rho(g)$. g is said to be of *null entropy* (resp. *positive entropy*) if $h(g) = 0$ (resp. $h(g) > 0$). For any subgroup $G \leq \mathrm{Aut}(X)$, the *null subset* $N(G)$ of G is the subset of G consisting of all automorphisms of null entropy; that is,

$$N(G) := \{g \in G \mid h(g) = 0\}.$$

A subgroup $G \leq \mathrm{Aut}(X)$ is of *null entropy* if every element $g \in G$ is of null entropy, i.e., if $G = N(G)$.

Remark. By the surface classification, a complex surface S has some $g \in \mathrm{Aut}(S)$ of positive entropy only if S is bimeromorphic to a rational surface, complex torus, K3 surface or their étale quotients (cf. [6, Proposition 1]). A similar phenomenon in higher dimension is discussed in [45].

We also note that in general $N(G)$ may not be a subgroup of G. The following is an example by Oguiso [30, §4].

Let E be an elliptic curve and X the minimal resolution of $(E \times E)/\{\pm 1\}$. Then the natural actions of

$$f_1' := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad f_2' := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on $E \times E$ extend to actions f_1, f_2 on the Kummer K3 surface X. As proved in [30], $G := \langle f_1, f_2 \rangle \cong \mathrm{SL}_2(\mathbb{Z})/\{\pm I_2\} = \mathrm{PSL}_2(\mathbb{Z})$ contains subgroups:

$$G_m := \langle f_1^m, f_2^m \rangle = \langle f_1^m \rangle * \langle f_2^m \rangle = \mathbb{Z} * \mathbb{Z}$$

for all $m \geq 2$. We can verify that $f_1^m, f_2^m \in N(G_m)$ and $N(G_m) = \{g \in G_m : |\operatorname{tr}(g)| \leq 2\}$. So $N(G_m)$ is not a subgroup of G_m.

Definition 3.2. A compact Kähler manifold (resp. projective) is *ruled* if it is bimeromorphic to a compact Kähler (resp. projective) manifold with a $\mathbb{P}^1$-fibration.

Remark. In particular, if the linear part of $\mathrm{Aut}_0(X)$ is nontrivial, then X is ruled (cf. [9, Proposition 5.10]).

Definition 3.3. A group G is *virtually nilpotent* (resp. *virtually abelian*, or *virtually abelian of rank* r) if there is a finite-index subgroup $G_1 \le G$ such that G_1 is nilpotent (resp. abelian, or isomorphic to $\mathbb{Z}^{\oplus r}$).

Definition 3.4. A group G is *virtually solvable* (resp. *almost abelian*, or *almost abelian of rank* r) if there is a finite-index subgroup $G_1 \le G$ and an exact sequence of groups

$$1 \to H \to G_1 \to Q \to 1$$

such that H is finite and Q is solvable (resp. abelian, or isomorphic to $\mathbb{Z}^{\oplus r}$).

Remark. In Definition 3.4, consider the conjugate action of G_1 on H. Since H is finite, we can replace G_1 and H by their finite-index subgroups such that the conjugate action of G_1 on H is trivial. In particular, H is abelian. So in the definition of virtually solvable group, we may further assume that $H = 1$, so that our definition coincides with the usual definition.

Definition 3.5. [44, §2.2] Suppose X is a compact Kähler (resp. projective) manifold of dimension n. Let $\bar{P}(X)$ be the closure of the Kähler cone (resp. the nef cone). The elements of $\bar{P}(X)$ are called *nef*. A sequence $L_1, \ldots, L_k$ $(1 \le k \le n-1)$, where $L_k \in H^{1,1}(X, \mathbb{R})$, is called *quasi-nef* if

(i) $0 \neq L_1 \in \bar{P}(X)$, and

(ii) for every $j = 1, \ldots, k$, there exist a nef sequence $(M_j(i))_{i=1}^{\infty}$ such that

$$0 \neq L_1 \cdots L_j = \lim_{i \to \infty} L_1 \cdots L_{j-1} M_j(i)$$

A group $G \leq \mathrm{Aut}(X)$ is said to be *polarized* by the quasi-nef sequence $L_1, \ldots, L_{n-1}$ if for there exist some characters $\chi_j : G \to (\mathbb{R}_{>0}, \times)$, $j = 1, \ldots, n-1$, such that

$$g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g) L_1 \cdots L_k$$

for all $g \in G$ and $k = 1, \ldots, n-1$.

Definition 3.6. Let g be an endomorphism of a variety X. A point $x \in X$ is called *g-preperiodic* if the g-orbit $\{g^s(x) \mid s \in \mathbb{N}\}$ is finite. The set of all g-preperiodic points is denoted by $\mathrm{Prep}(g)$.

Let $G \leq \mathrm{Aut}(X)$ be a subgroup. A subvariety $Z \subseteq X$ is *G-periodic* if there is a finite-index subgroup G_1 of G such that $g(Z) = Z$ for all $g \in G_1$.

Definition 3.7. Let $G \leq \mathrm{Aut}(X)$ be a subgroup of automorphisms. (X, G) is called *non-strongly primitive* if there is a surjective holomorphic map $X' \to Y$ with $0 < \dim Y < \dim X$ for some Kähler manifold X' bimeromorphic to X, such that for some finite-index subgroup G_1 of G, the induced action of G_1 on X' is regular and descends to an action on Y with $X' \to Y$ being G_1-equivariant. (X, G) is said to be *strongly primitive* if it is not non-strongly primitive.

3.3 Proofs of Theorems

We use the terminology and notations in [20], and the book of Hartshorne [13].

3.3.1 Lemmas

Lemma 3.8. *With the notations in Theorem 3.1, a subgroup $G \leq \mathrm{Aut}(X)$ has a finite restriction $G|_{L_{\mathbb{C}}}$ if and only if the index $|G : G \cap \mathrm{Aut}_0(X)|$ is finite.*

Proof. Consider the exact sequence

$$1 \to K \to G \to G|_{L_{\mathbb{C}}} \to 1.$$

Let ω be a Kähler class of X (or an ample divisor if X is projective). Define

$$\mathrm{Aut}_\omega(X) := \{g \in \mathrm{Aut}(X) \mid g^*\omega = \omega\}.$$

Then $K \leq \mathrm{Aut}_\omega(X)$, and by [21, Proposition 2.2] $\mathrm{Aut}_0(X) \lhd \mathrm{Aut}_\omega(X)$ is finite-index. Now

$$K/(K \cap \mathrm{Aut}_0(X)) \cong (K \,\mathrm{Aut}_0(X))/\,\mathrm{Aut}_0(X) \leq \mathrm{Aut}_\omega(X)/\,\mathrm{Aut}_0(X),$$

it follows that $K/(K \cap \mathrm{Aut}_0(X))$ is finite. Note that $\mathrm{Aut}_0(X)$ acts trivially on $H^2(X, \mathbb{Z})$ (resp. $\mathrm{NS}(X)$), and hence on $L_{\mathbb{C}}$. We have $G \cap \mathrm{Aut}_0(X) = K \cap \mathrm{Aut}_0(X)$. Since

$$|G : G \cap \mathrm{Aut}_0(X)| = |G : K|\,|K : K \cap \mathrm{Aut}_0(X)|,$$

$|G : G \cap \mathrm{Aut}_0(X)|$ is finite if and only if $G|_{L_{\mathbb{C}}} \cong G/K$ is finite. This completes the proof of Lemma 3.8. □

Lemma 3.9. *Consider the exact sequence of groups*

$$1 \to N \to G \to Q \to 1$$

with N contained in a (not necessarily connected) affine algebraic group, and G contained in a (possibly infinite) union of affine algebraic groups. Suppose that both N and Q are virtually solvable. Then so is G.

Proof. Let $\bar{N}$ be the Zariski-closure in its algebraic over group. Replacing G by a finite-index subgroup if necessary, we may assume that Q is solvable. Since $\bar{N} \cap G \lhd G$, we have $(\bar{N})_0 \cap G \lhd G$, where $(\bar{N})_0$ is the identity component of $\bar{N}$. Define $M :=(\bar{N})_0 \cap G$. Then we have an exact sequence

$$1 \to N/M \to G/M \to G/N = Q \to 1.$$

Note that

$$N/M \cong (N(\bar{N})_0)/(\bar{N})_0 \leq (\bar{N})/(\bar{N})_0,$$

and $(\bar{N})/(\bar{N})_0$ a finite group. Then N/M is also finite. Replacing G/M by a finite-index subgroup, we may assume that the conjugate action of G/M on N/M is trivial. In particular, N/M is abelian; so G/M is solvable.

Since N is virtually solvable, $\bar{N}$ is also virtually solvable. Hence $(\bar{N})_0$ is solvable, and so is M. Therefore, G is solvable. □

For a complex torus X (as a variety), we have $\mathrm{Aut}_{\mathrm{variety}}(X) = T \rtimes \mathrm{Aut}_{\mathrm{group}}(X)$ where $T = \mathrm{Aut}_0(X)$ ($\cong X$) consists of all the translations of X and $\mathrm{Aut}_{\mathrm{group}}(X)$ is the group of group automorphisms on X.

Lemma 3.10. *Let X be a compact complex torus of dimension n, and $\mathrm{Aut}_0(X) \leq G \leq \mathrm{Aut}_{\mathrm{variety}}(X)$. Then $G/\mathrm{Aut}_0(X)$ is almost abelian if and only if it is virtually abelian.*

Proof. Note that $H := G/\operatorname{Aut}_0(X)$ can be naturally embedded into the general linear group $\mathrm{GL}(H^0(X, \Omega_X^1))$. Let $\bar{H}$ be its Zariski-closure, and $(\bar{H})_0$ the identity component of $\bar{H}$.

The "only if" part is clear (cf. Definition 3.3, 3.4). For the "if" part, suppose that H is almost abelian. Replacing H by a finite-index subgroup, we may assume that H has a finite normal subgroup H_1 such that H/H_1 is abelian. Note that H_1 is also normal in $\bar{H}$. Consider the exact sequence

$$1 \to H_1 \to \bar{H} \to \bar{H}/H_1 \to 1.$$

Since the abelian group H/H_1 is Zariski-dense in $\bar{H}/H_1$, the latter is also abelian. So the connected commutator group $[(\bar{H})_0, (\bar{H})_0]$ is contained in the finite group H_1, and hence it is trivial. Thus, $(\bar{H})_0$ is abelian. Now $H \cap (\bar{H})_0$ is abelian and of finite-index in H. So H is virtually abelian. This proves the lemma. $\square$

Lemma 3.11. *Let G be a group, $H \lhd G$ a finite normal subgroup, and $g_1, g_2 \in G$. Suppose that $\bar{g}_1 = \bar{g}_2$ in G/H. Then there exists a positive integer s such that $g_1^s = g_2^s$.*

Proof. By the assumption, $g_1^n g_2^{-n} \in H$ for all $n \in \mathbb{Z}$. Since H is finite, there exist integers $m < n$ such that $g_1^m g_2^{-m} = g_1^n g_2^{-n}$. Then $g_1^s = g_2^s$ by taking $s := n - m$. $\square$

Lemma 3.12 (Tits alternative, [35, Theorem 1])**.** *Over a field of characteristic* 0*, a linear group either has a non-abelian free subgroup or possesses a solvable subgroup of finite index.*

3.3.2 Tits Type Theorems for Manifolds

In this section, we are going to give the proofs of Theorem 3.1–3.3 and some related results of Tits type of compact Kähler (resp. projective) manifolds of dimension n.

Proposition 3.13 mostly follows from Oguiso [29, Lemma 2.5] and Tits (Lemma 3.12).

Proposition 3.13. *Let X be a compact Kähler (resp. projective) manifold of dimension n, and $G \le \mathrm{Aut}(X)$ a subgroup. With the notations in Theorem 3.1, we have the following assertions.*

1) *Suppose that G is of null entropy. Then $G|_{L_{\mathbb{C}}}$ is virtually unipotent (and hence virtually solvable). Moreover, $G|_{L_{\mathbb{C}}}$ is finitely generated.*

2) *Suppose that $G|_{L_{\mathbb{C}}} \ge \mathbb{Z} * \mathbb{Z}$. Then G contains an element of positive entropy.*

Proof. (2) follows immediately from (1); so we will only prove (1). Suppose that G is of null entropy. By following the proof of [29, Lemma 2.5], the subset

$$U := \{g \in G : g|_{L_{\mathbb{C}}} \text{ is unipotent}\}$$

is a normal subgroup of G. If $G|_{L_{\mathbb{C}}}$ is not virtually solvable, then by the classical Tits alternative theorem (Lemma 3.12) $G|_{L_{\mathbb{C}}} \ge \mathbb{Z} * \mathbb{Z}$. So there exist $g_1, g_2 \in G$ such that

$$\langle g_1, g_2\rangle|_{L_{\mathbb{C}}} = (\langle g_1\rangle|_{L_{\mathbb{C}}}) * (\langle g_2\rangle|_{L_{\mathbb{C}}}) = \mathbb{Z} * \mathbb{Z}.$$

By [29, Proposition 2.2(2)], there is a positive integer s such that $g^s \in U$ for all $g \in G$. In particular,

$$\mathbb{Z} * \mathbb{Z} = \langle g_1^s, g_2^s\rangle|_{L_{\mathbb{C}}} \le U|_{L_{\mathbb{C}}}$$

which is unipotent, and hence solvable. This is absurd.

Therefore, $G|_{L_\mathbb{C}}$ is virtually solvable. Replacing G by a finite-index subgroup if necessary, we may assume that $G|_{L_\mathbb{C}}$ is solvable and its closure, denoted by $\bar{G}$, in $\mathrm{GL}(L_\mathbb{C})$ is connected. Then $\bar{G}$ is also solvable.

Let $\bar{U}$ be the unipotent radical of $\bar{G}$. By [29, Proposition 2.2(2)] again, there exists a positive integer n such that $g^n|_{L_\mathbb{C}} \in \bar{U}$ for all $g \in G$. Then the image of G via the composition

$$G \to G|_{L_\mathbb{C}} \to \bar{G} \to \bar{G}/\bar{U}$$

is a subgroup of $\mathrm{GL}(L_\mathbb{C})$ of finite exponent; hence it is a finite group by Burnside's theorem. Note that U is the kernel of $G \to \bar{G}/\bar{U}$. Then $|G : U|$ is finite. Therefore, G is virtually unipotent.

We may further assume that $G = U$. Let $\ell = \mathrm{rank}(L)$, where $L = H^2(X, \mathbb{Z})/(\text{torsion})$ (resp. $L = \mathrm{NS}(X)/(\text{torsion})$). Then as in [14, §19.1 or Exercise 17.1], there is a normal series

$$1 \lhd U_1 \lhd U_2 \lhd \cdots \lhd U_{\ell(\ell-1)/2} = \bar{G}$$

such that each factor group is of 1-dimensional. Restricting this series to $G|_L$, we get a normal series of discrete groups whose factor groups are cyclic. Thus, $G|_L$ is generated by at most $\ell(\ell-1)/2$ elements, and so is $G|_{L_\mathbb{C}}$. This proves Proposition 3.13. □

Proof of Theorem 3.1. Suppose that $G|_{L_\mathbb{C}}$ does not contain $\mathbb{Z} * \mathbb{Z}$. Then by the Tits alternative theorem (Lemma 3.12), $G|_{L_\mathbb{C}}$ is virtually solvable. Replacing G by a finite-index subgroup if necessary, we may assume that $G|_{L_\mathbb{C}}$ is solvable and its closure, denoted by $\bar{G}$, in $\mathrm{GL}(L_{\mathbb{C}})$ is connected. Consider the exact sequence

$$1 \to K \to G \to G|_{L_{\mathbb{C}}} \to 1$$

as in Lemma 3.8.

Suppose that K is virtually solvable. Then so is G by Lemma 3.9. By [44, §2.7], $\bar{G}$ is polarized (cf. Definition 3.5) by a quasi-nef sequence $L_1, \ldots, L_{n-1}$, and there is a homomorphism

$$\psi : G \to \mathbb{R}^{n-1}, \quad g \mapsto (\log \chi_1(g), \ldots, \log \chi_{n-1}(g)),$$

such that for all $g \in G$ and $1 \leq k \leq n-1$,

$$g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g) L_1 \cdots L_k,$$

where $\chi_j : G \to (\mathbb{R}_{>0}, \times)$ are characters. Further, by [44, Claim 2.8, 2.9] $\mathrm{Ker}(\psi) = N(G)$ and $\mathrm{Im}(\psi)$ is discrete in $\mathbb{R}^{n-1}$. Thus Theorem 3.1(3) occurs.

Suppose that K is not virtually solvable. Consider the exact sequence

$$1 \to L(X) \to \mathrm{Aut}_0(X) \to T \to 1$$

where $L(X)$ is the linear part of $\mathrm{Aut}_0(X)$ and T is a compact complex torus (cf. [9, Theorem 5.5]). This induces an exact sequence

$$1 \to K \cap L(X) \to K \cap \mathrm{Aut}_0(X) \to Q \to 1$$

where Q is abelian.

If Theorem 3.1(2) does not occur, then by the Tits alternative theorem (Lemma 3.12) $K \cap L(X)$ is virtually solvable. So it follows from Lemma 3.9 and the exact sequence above that $K \cap \mathrm{Aut}_0(X)$ also is virtually solvable. On the other hand, since K acts trivially on $L_{\mathbb{C}}$, by Lemma 3.8 $K/(K \cap \mathrm{Aut}_0(X))$ is a finite group. However, this would imply that K is also virtually solvable, contradicting our assumption.

For the final assertion, suppose $G|_{L_{\mathbb{C}}}$ is virtually solvable. Replacing G by a finite-index subgroup if necessary, we may assume that $G|_{L_{\mathbb{C}}}$ is solvable and its Zariski-closure in $\mathrm{GL}(L_{\mathbb{C}})$ is connected. Then by [44, Theorem 1.2], the null set $N(G)$ is a normal subgroup of G such that $G/N(G) \cong \mathbb{Z}^{\oplus r}$ for some $r \leq n-1$. Applying Proposition 3.13 to $N(G)$, $N(G)|_{L_{\mathbb{C}}}$ is finitely generated. Thus the assertion follows. This completes the proof of Theorem 3.1. $\square$

Proof of Theorem 3.2. (1) is proved in [44, §2.7].

(2) Suppose that G is polarized by a quasi-nef sequence $L_1, \ldots, L_{n-1}$. Then $g^*(L_1 \cdots L_k) = \chi_1(g) \cdots \chi_k(g) L_1 \cdots L_k$ for all $g \in G$, where $\chi_j : G \to (\mathbb{R}_{>0}, \times)$ are characters. As in the proof of [44, Theroem 1.2], define the homomorphism

$$\psi : G \to \mathbb{R}^{n-1}, \quad g \mapsto (\log \chi_1(g), \ldots, \log \chi_{n-1}(g)).$$

It follows from [44, Claim 2.8, 2.9] that $\mathrm{Ker}(\psi) = N(G)$ and $\mathrm{Im}(\psi) \cong \mathbb{Z}^{\oplus r}$ is a lattice in $\mathbb{R}^{n-1}$. Consider the exact sequence

$$1 \to N(G) \to G \to \mathbb{Z}^{\oplus r} \to 1$$

and its induced exact sequence

$$1 \to N(G)|_{L_{\mathbb{C}}} \to G|_{L_{\mathbb{C}}} \to Q \to 1,$$

Since $N(G)$ is of null entropy, by Proposition 3.13, $N(G)|_{L_{\mathbb{C}}}$ is virtually solvable. Applying Lemma 3.9 to the exact sequence above, we see that $G|_{L_{\mathbb{C}}}$ is also virtually solvable.

(3) Suppose $G|_{L_{\mathbb{C}}}$ is virtually solvable. Then G has a finite-index subgroup G_1 such that $G_1|_{L_{\mathbb{C}}}$ is solvable and its Zariski-closure in $\mathrm{GL}(L_{\mathbb{C}})$ is connected. By [44, Theorem 1.2], we have $N(G_1) \lhd G_1$. Note that G_1 is normal in G. Then for any $g \in G$,

$$gN(G)g^{-1} \subseteq (gG_1g^{-1}) \cap N(G) = G_1 \cap N(G) = N(G_1).$$

Thus $N(G_1)$ is normal in G. This completes the proof. $\square$

3.3.3 Projective Surfaces

In this section, we work on projective surfaces.

Proof of Theorem 3.4. For the "if" part, suppose there is a G-equivariant fibration $X \to B$. Let F be a fiber. Then for any $g \in G$, $g^*F \equiv F$, and it follows from [43, Lemma 2.12] that g is of null entropy.

For the "only if" part, suppose that $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is infinite and G is of null entropy. Then X is not of general type, and hence the Kodaira dimension $\kappa(X) \leq 1$. By [29, Lemma 2.8] or [43, Lemma 4.3], there is a unique (up to scalar multiplication) nonzero pseudo-effective divisor L such that $L^2 = 0$ and $g^*L \equiv L$ for all $g \in G$; furthermore, L is nef and it can be chosen to be integral.

If $\kappa(X) = 1$, or X is a hyperelliptic surface, or X is an irrational ruled surface, then X has a typical fibration which is clearly G-equivariant.

It remains to consider (up to blowups of) K3, Enriques, abelian and rational surfaces. We may assume that X is minimal unless it is rational.

i) If X is a K3 surface, then the integral divisor L chosen above is parallel to a fiber of an elliptic fibration.

ii) If X is an Enriques surface, its canonical cover (cf. Section 2.1) is a K3 surface. Consider the pullback of L. The corresponding fibration on the K3 cover descends to a G-equivariant fibration on X as required.

iii) Suppose that X is a rational surface. We may assume that (X, G) is minimal, that is, there is no nonempty finite set of disjoint (-1)-curves on X which is G-stable (cf. [43, Definition 2.2]). Then by [43, Theorem 4.1], L is parallel to $-K_X$, and it follows from [10, Theorem 2] that there exists a natural number m such that $|-mK_X|$ defines an elliptic fibration, which is clearly G-equivariant.

iv) Suppose that X is an abelian surface. Since $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is infinite, as in [29, Proposition 2.2] by Burnside's theorem, there exists some $g \in G$ such that $g|_{\mathrm{NS}_{\mathbb{C}}(X)}$ has infinite order. On the other hand, since g is of null entropy, after replacing g by its power if necessary, we may assume that $g^*|_{H^0(X,\Omega^1_X)}$ is unipotent. Let F be a 1-dimensional component in $\mathrm{Ker}(g - \mathrm{id}_X)$. Then $g^*F = F$. If L and F are not parallel, then the class of $L + F$ is big and fixed by g^*; it follows from [21, Proposition 2.2] or [45, Lemma 2.23] that $g^s \in \mathrm{Aut}_0(X)$ for some $s > 0$. In particular, $g|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is of finite order, a contradiction. Therefore, we may assume that $L = F$ whose class is fixed by G. Now the quotient map $X \to X/F$ is a G-equivariant fibration.

This completes the proof of Theorem 3.4. $\square$

Proposition 3.14. *Let X be a smooth projective surface and $G \leq \mathrm{Aut}(X)$ a subgroup. Suppose that the subset $N(G)$ is a subgroup of G and $N(G) \neq G$. Then we have the following assertions:*

1) *$N(G)|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is a finite group.*

2) *Suppose that $N(G)$ is an infinite group. Then X is a complex torus, $H := N(G) \cap \mathrm{Aut}_0(X)$ $(= G \cap \mathrm{Aut}_0(X))$ is Zariski-dense in $\mathrm{Aut}_0(X)$ $(\cong X)$, and the index $|N(G) : H| < \infty$.*

Remark. If a projective surface X has $\mathrm{Aut}_0(X) \neq 1$ and an automorphism g of positive entropy, then by applying Proposition 3.14 to $G := \mathrm{Aut}_0(X)\langle g\rangle$ we see that X is birational to an abelian surface. So Proposition 3.14 is a sort of a clean version of Theorem 3.6 when $\dim X = 2$.

Proof of Proposition 3.14. As in the proof of Theorem 3.2, replacing G by a finite-index subgroup if necessary, we may assume that the Zariski-closure of $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ in $\mathrm{GL}(\mathrm{NS}_{\mathbb{C}}(X))$ is connected.

(1) Suppose to the contrary that $N(G)|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is infinite. As in the proof of Theorem 3.3, there is a unique (up to scalar multiplication) nonzero nef divisor L such that $h^*L \equiv L$ for all $h \in N(G)$; furthermore, we may choose L to be a $\mathbb{Q}$-

divisor. Since $N(G) \lhd G$, for every $g \in G$

$$h^*(g^*L) = g^*(g^{-1}hg)^*L \equiv g^*L.$$

By uniqueness, g^*L is parallel to L for all $g \in G$. Then by [43, Lemma 2.12], every $g \in G$ is of null entropy, contradicting our assumption that $N(G) \neq G$.

(2) Since $N(G) \neq G$, the Kodaira dimension $\kappa(X) \leq 0$ by [45, Lemma 2.13]. Since $N(G)|_{L_{\mathbb{C}}}$ is finite, by Lemma 3.8, $H := N(G) \cap \mathrm{Aut}_0(X)$ has finite index in $N(G)$. In particular, $\mathrm{Aut}_0(X) \neq 1$. So X is neither a K3 or an Enriques surface. Suppose that X is a rational surface. By [12, Proposition (1.3)], X has only finitely many (-1)-curves. Let $g \in G \backslash N(G)$, and replace g by its power if necessary, we may assume that g acts regularly on a relatively minimal model X_m of X. Then $(g^2)^*$ fixes a nonzero nef class on X_m, and thus it is of null entropy (cf. [43, Lemma 2.12]). This is absurd.

By Theorem 3.4, the assumption $G \neq N(G)$ implies that X has no typical fibration preserved by G or a finite-index subgroup, and hence X must be birational to an abelian surface. It remains to show that the albanese map $\mathrm{alb}_X : X \to A := \mathrm{Alb}(X)$ is an isomorphism. By [44, Lemma 2.13], alb_X is surjective, birational, and necessarily $\mathrm{Aut}(X)$-equivariant.

Let G_0 be the Zariski-closure of H in $\mathrm{Aut}_0(X)$. Then G_0 is normalized by G. It follows from [44, Lemma 2.14] that some G_0-obit is dense in X, i.e., X is dominated by G_0. Then A is also dominated by $G_0|_A$. We have $G_0|_A = \mathrm{Aut}_0(X)$ $(\cong A)$.

Suppose to the contrary that alb_X is not an isomorphism and we let $B \subset A$ be the locus over which alb_X is not isomorphic. Note that B and $\mathrm{alb}_X^{-1}(B)$ are G_0-stable. Since $G_0|_A$ consists of all translations on A, which has no fixed point, every irreducible component B_i of B must be of dimension 1. Replacing G by a finite-index subgroup if necessary, we may assume that each B_i is stabilized by G. If some B_i is of general type, then $G_0|_{B_i} \leq \mathrm{Aut}(B_i)$ is a finite group, and hence $G_0|_{B_i} = \{\mathrm{id}\}$. But this is impossible because the translation group $G|_{A_0}$ has no fixed point. Thus B_i is not of general type. As in the proof of [48, Lemma 2.11, Case(2)(3)], A (and hence X) has a non-trivial G-equivariant fibration. This contradicts the assumption that $N(G) \neq G$ by Theorem 3.4.

Therefore, X is a complex torus; and we complete the proof of Proposition 3.14.

$\square$

Proposition 3.15. *Let X be a smooth projective surface and $G \leq \mathrm{Aut}(X)$ a subgroup. Suppose that $G \neq N(G)$, and $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is solvable and Z-connected (i.e., its Zariski-closure in $\mathrm{GL}(\mathrm{NS}_{\mathbb{C}}(X))$ is connected). Then we have the following assertions:*

1) *$N(G)$ is normal in G, and $G/N(G) \cong \mathbb{Z}$, whence $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is almost abelian of rank 1.*

2) *Suppose further that X is not a complex torus. Then G is almost abelian of rank 1.*

Proof. For (1), $N(G) \lhd G$ and $G/N(G) \cong \mathbb{Z}$ follow from [44, Theorem 1.2]. The last assertion follows from Proposition 3.14(1).

For (2), if X is not a torus, then by Proposition 3.14(2), $N(G)$ is finite. Since $G/N(G) \cong \mathbb{Z}$, G is almost abelian of rank 1. $\square$

Proof of Theorem 3.3. Suppose $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ does not contain $\mathbb{Z} * \mathbb{Z}$. By [44, Theorem 1.1], there is a finite-index subgroup G_1 of G such that $G_1|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is solvable and Z-connected.

If G_1 is of null entropy, so is G. Hence by [29, Theorem 2.1], $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is almost abelian of rank $r \leq \max\{1, \operatorname{rank} \mathrm{NS}_{\mathbb{Q}}(X) - 2\}$. So there is a finite-index subgroup $G_2 \leq G$ and a normal subgroup $H_2 \lhd G_2$ such that $G_2/H_2 \cong \mathbb{Z}^{\oplus r}$ and $H_2|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is finite. Then by Lemma 3.8, $|H_2 : H_2 \cap \mathrm{Aut}_0(X)| < \infty$. In particular, if $\mathrm{Aut}_0(X) = 1$, then H_2 is a finite group, and hence G is almost abelian of rank r.

Suppose $G_1 \neq N(G_1)$. Recall that $|G : G_1| < \infty$. Then $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is almost abelian of rank 1 by Proposition 3.15(1). Suppose further that $\mathrm{Aut}_0(X) = 1$. Then X is not a torus (otherwise $X \cong \mathrm{Aut}_0(X) = 1$), and hence G is almost abelian by Proposition 3.15(2). $\square$

Proof of Theorem 3.5. Since $g_1 \in \mathrm{Aut}(X)$ is of positive entropy, we may assume that X is a K3, Enriques, abelian or rational surface, and X is minimal unless it is rational.

Let $G := \langle g_1, g_2 \rangle$. Then by [43, Theorem 3.1], we can decomposed $G = \langle h \rangle \ltimes C$, where $C|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is finite. Write $g_1 = h^{n_1} t_1$ and $g_2 = h^{n_2} t_2$ ($t_1, t_2 \in C$). Then $(\bar{g}_1)^{n_2} = (\bar{g}_2)^{n_1}$ in G/C. By restricting to $\mathrm{NS}_{\mathbb{C}}(X)$ and applying Lemma 3.11, we have $g_1^{n_2 s} = g_2^{n_1 s}$ for some $s > 0$ in $\mathrm{Aut}(X)|_{\mathrm{NS}_{\mathbb{C}}(X)}$. This proves Theorem 3.5(1).

For (2), let H' be an ample divisor and set $H := \sum_{g^* \in C|_{\mathrm{NS}_{\mathbb{C}}(X)}} g^* H'$. Then

$$C \leq \mathrm{Aut}_H(X) := \{g \in \mathrm{Aut}(X) \mid g^* H \equiv H\}.$$

By [21, Proposition 2.2] or [9, Theorem 4.8], $|\mathrm{Aut}_H(X) : \mathrm{Aut}_0(X)| < \infty$. It follows that $|C : C \cap \mathrm{Aut}_0(X)| < \infty$.

If $\mathrm{Aut}_0(X) = 1$ (this is the case for K3 or Enriques surface), then C is finite. Applying Lemma 3.11 to $C \lhd G$, we see that $g_1^{n_2 t} = g_2^{n_1 t}$ for some $t > 0$ in $\mathrm{Aut}(X)$.

Suppose that $\mathrm{Aut}_0(X) \neq 1$. By following the proof of Proposition 3.14(2), we see that X cannot be a rational surface.

Finally, we assume that X is an abelian surface and $\mathrm{Prep}(g_1) \cap \mathrm{Prep}(g_2) \neq \emptyset$. Replacing g_1 and g_2 by some common power, we may assume that they fix a point $x_0 \in X$. Note that for any $s \in \mathbb{N}$, we have $c_s := g_1^s g_2^{-s} \in C \leq \mathrm{Aut}_H(X)$ satisfies $c_s(x_0) = x_0$; so c_s are polarized by H, which are known to form a finite group. Then it follows from the proof of Lemma 3.11 that $g_1^s = g_2^s$ for some $s > 0$. This proves Theorem 3.5(2). $\square$

3.3.4 Projective Threefolds

In this section, we will prove Theorem 3.6 in several steps.

Proof of Theorem 3.6. We allow X to have terminal singularities. Replacing G_0 by the identity connected component of its Zariski-closure in $\mathrm{Aut}_0(X)$ (and also replacing G), we may assume that $G_0 = G \cap \mathrm{Aut}_0(X)$ is connected, positive-dimensional and closed in $\mathrm{Aut}_0(X)$. As in the proof of [44, Lemma 2.14], if X is not dominated by G_0, then there would be a G-equivariant quotient map $\pi : X \dashrightarrow Y := X/G_0$ with $0 < \dim Y < 3$; contradicting the assumption that (X, G) is strongly primitive. Therefore, X is homogeneous and dominated by G_0.

Claim 3.16. *Suppose that $q(X) > 0$. Then Theorem 3.6 is true.*

Proof. As in the proof of [44, Lemma 2.13], since (X, G) is strongly primitive, the albanese map $\mathrm{alb}_X : X \to A := \mathrm{Alb}(X)$ is surjetive, birational and $\mathrm{Aut}(X)$-equivariant. By using the same argument as in the proof of Proposition 3.14(2) (here each B_i is of dimension 1 or 2), we see that alb_X is an isomorphism. This proves Claim 3.16. □

We continue the proof of Theorem 3.6. By Claim 3.16, we may assume that $q(X) = 0$. Then $G_0 \leq \mathrm{Aut}_0(X)$ is a linear algebraic group dominating X (cf. [21, Theorem 3.12], [9, Theorem 5.5]). By the classic result of Chevalley, linear algebraic groups are rational varieties. In particular, X is a ruled and uniruled variety.

Let $U \subseteq X$ be an open dense G_0-orbit and denote $F := X \backslash U$. Then F consists of finitely many prime divisors and subvarieties of codimension ≥ 2. Since $G_0 \lhd G$, U is G-stable. After replacing G by a finite-index subgroup, we may further assume that all the irreducible components of F are G-stable. In particular, there are finitely many G_0-periodic prime divisors on X, which are contained in F; hence they are G-stable.

Claim 3.17. 1) *Every G_0-periodic subvariety of X is G_0-stable.*

2) *There is a composite* $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m$ *of birational extremal contractions and an extremal Fano contraction* $X_m \to Y$ *with* $\dim Y < \dim X$*. The induced action of* G_0 *on each* X_i *is regular.* $G_0|_{X_m}$ *descends to an action on* Y *so that* $X_m \to Y$ *is* G_0*-equivariant.*

3) *In (2), for every finite-index subgroup* G_1 *of* G*, there is at least one* $i \in \{1, \dots, m\}$ *such that the induced action of* G_1 *on* X_i *is not regular.*

4) *In (2), let* $s \leq m$ *be the largest integer such that* $X_{i-1} \to X_i$ *is divisorial for every* $i < s$*. Then, replacing* G *by its finite-index subgroup, the induced action of* G *on each* X_i $(0 \leq i \leq s-1)$ *is regular and hence each map* $X_{i-1} \to X_i$ *is* G*-equivariant. In particular,* $s < m$*.*

Proof. (1) This is true because G_0 is a continuous group.

(2) Since X is uniruled, the Kodaira dimension $\kappa(X) = -\infty$ and the existence of the sequence follows from the MMP (cf. [5]). Note that G_0 acts trivially on $H^i(X, \mathbb{Z})$, $\mathrm{NS}_{\mathbb{C}}(X)$ and the extremal rays of $\overline{\mathrm{NE}}(X)$. The second assertion follows by induction (cf. [47, Lemma 2.12, 3.6]).

(4) Suppose that $X \to X_1$ is a divisorial contraction of an extremal ray $\mathbb{R}_{\geq 0}[\ell]$ with an exceptional divisor D_0. Since G_0 acts trivially on the extremal rays of $\overline{\mathrm{NE}}(X)$, D_0 must be G_0-stable. In particular, $G_0 \subseteq F$, and thus it is G-stable. Recall that G/G_0 is almost abelian of rank $r > 0$. Replacing G by a finite-index subgroup if necessary, there is a normal subgroup G_1 of G such that $|G_1 : G_0| < \infty$ and $G/G_1 = \langle \bar{g}_1 \rangle \oplus \cdots \oplus \langle \bar{g}_r \rangle$. By [47, Lemma 3.7], $X \to X_1$ is $g_i^{s_i}$-equivariant for some

$s_i > 0$. Replacing g_i by its powers (also G by its finite-index subgroup), we may assume that $X \to X_1$ is g_i-equivariant. Since $g_i\ell \equiv \ell$ for all i, $\{g\ell \mid g \in G\}$ consists only a finite number ($\le |G_1 : G_0|$) of equivalence classes. Therefore, the class of ℓ is fixed by a finite-index subgroup of G.

(3) Replacing G by its finite-index subgroup, we may suppose to the contrary that G acts regularly on all X_i. As in the proof of (4) above, applying [47, Theorem 2.13 or Appendix], we may assume that $X_m \to Y$ is G-equivariant. Since (X, G) is strongly primitive, we must have $\dim Y = 0$ and hence $\rho(X_m) = 1$. On the other hand, as in the proof of [44, Lemma 2.12], the strongly primitivity of (X, G) implies that the anti-Kodaira dimension $\kappa(X_m, -K_{X_m}) \le 0$. This is absurd because $-K_{X_m}$ is ample. □

Claim 3.18. *With the notations in Claim 3.17, it is impossible that* $\mathrm{NS}_{\mathbb{C}}(X_i)$ *($0 \le i \le m$) is spanned by $-K_{X_i}$ and G_0-periodic divisors, or* $\mathrm{NS}_{\mathbb{C}}(Y)$ *is spanned by G_0-periodic divisors.*

Proof. Note that $\mathrm{NS}_{\mathbb{C}}(X_m)$ is spanned by $-K_{X_m}$ and the pullback of $\mathrm{NS}_{\mathbb{C}}(Y)$, and that $\mathrm{NS}_{\mathbb{C}}(X)$ is spanned by the pullback of $\mathrm{NS}_{\mathbb{C}}(X_i)$ and the exceptional divisors of $X \dashrightarrow X_i$. So we only need to rule out the possibility that $\mathrm{NS}_{\mathbb{C}}(X)$ is spanned by $-K_X$ and G_0-stable divisors D_i, all of which are contained in F and hence G-stable.

Let H be an ample divisor on X. Then it can be written as a combination of $-K_X$ and D_i's. In particular, $G \le \mathrm{Aut}_H(X)$. Then as in the proof of Lemma 3.8 we see that $|G : G_0| < \infty$. This contradicts our assumption. □

Claim 3.19. *X_m and hence Y contain a G_0-fixed point.*

Proof. If $X_{m-1} \dashrightarrow X_m$ is a flip, then $\operatorname{Sing} X_{m-1} \neq \emptyset$ because a smooth threefold has no flip, and hence $\operatorname{Sing} X_m \neq \emptyset$ because a flip preserves the singular type of varieties. Then the isolated points in $\operatorname{Sing} X_m$ are fixed by G_0.

As in Claim 3.17(4), if $X_{t-1} \dashrightarrow X_t$ is a flip for some t and $X_t \to \cdots \to X_m$ is the composite of extremal divisorial contractions, then the isolated points in $\operatorname{Sing} X_{t+1}$ and hence their images on X_m are fixed by G_0. □

Claim 3.20. *It is impossible that* $\dim Y \leq 1$.

Proof. If $\dim Y = 0$, then $\rho(X_m) = 1$ and $\mathrm{NS}_{\mathbb{C}}(X_m)$ is spanned by $-K_{X_m}$. But this contradicts Claim 3.18. If $\dim Y = 1$, then $\rho(X_m) = 2$ and $\mathrm{NS}_{\mathbb{C}}(X_m)$ is spanned by $-K_{K_m}$ and the fiber over a G_0-fixed point (cf. Claim 3.19), contradicting Claim 3.18 again. □

We now continue the proof of Theorem 3.6. Let $\mathbb{R}_{\geq 0}[\ell]$ be an extremal ray on X_s (cf. Claim 3.17(2)) generated by a rational curve ℓ, and $X_s \dashrightarrow X_{s+1}$ the flip. Let $X_s \to Y_s$ be the flipping contraction. Then all the irreducible components E_i of its exceptional locus is stabilized by G_0. Replacing G be a finite-index subgroup if necessary, we may assume that G stabilizes all the irreducible components D_{ij} of the Zariski-closure of $\bigcup_{g \in G} g(E_i)$. These D_{ij} are unions of "small" G_0-orbits, and hence they are contained in the image of F.

i) If $\dim D_{ij} = \dim E_i = 1$, then G preserves the extremal ray $\mathbb{R}_{\geq 0}[\ell] \subseteq \overline{\mathrm{NE}}(X_s)$.

It follows from [47, Lemma 3.6] that G can be descended to a regular action on X_{s+1}. Now apply MMP on X_{s+1} and continue the process.

ii) Assume that $\dim D_{ij} = 2 > \dim E_i = 1$. Suppose G_0 acts trivially on $g_0(E_i)$ for some $g_0 \in G$. Since $G_0 \lhd G$, similarly as in the proof of Proposition 3.14(1), G_0 must act trivially on $g(E_i)$ for all $g \in G$. It thus follows that $G_0|_{D_{ij}} = \{\mathrm{id}\}$. This contradicts Claim 3.21 below.

Suppose that G_0 acts non-trivially on all $g(E_i)$ $(g \in G)$. Then $\{g(E_i) \mid g \in G\}$ are fibers of the quotient map $D_{ij} \to D_{ij}/G_0$. Hence, they give rise to the same class in the extremal ray $\mathbb{R}_{\geq 0}[\ell] \subseteq \overline{\mathrm{NE}}(X_s)$. In particular, G preserves this extremal ray. So by [47, Lemma 3.6] again, we can descend G to a regular action on X_{s+1}. Now apply MMP on X_{s+1} and continue the process.

Claim 3.21. *It is impossible that* $\dim Y = 2$, $\dim D_{ij} = 2$ *and* $G_0|_{D_{ij}} = \{\mathrm{id}\}$.

Proof. Note that $X_m \to Y$ is an extremal conic fibration. We can G_0-equivariantly resolve the indeterminacy of $\pi_s : X_s \dashrightarrow X_m \to Y$. By the proof of [23, Theorem 4.8], there is a an extremal conic fibration $\pi' : X' \to Y'$ with X', Y' smooth, and birational morphisms $\sigma_x : X' \to X_s$ and $\sigma_y : Y' \to Y$ such that $\pi_s \circ \sigma_x = \sigma_y \circ \pi'$. Note that G_0 stabilizes the extremal rays, we may also assume that these four maps are G_0-equivariant by taking equivariant blowups in the construction.

If $(K_{Y'})^2 \leq 7$, then $\mathrm{NS}_{\mathbb{C}}(Y')$ are spanned by the negative curves on Y', which are G_0-stable. This would imply that $\mathrm{NS}_{\mathbb{C}}(Y)$ is also spanned by G_0-stable curves (i.e., G_0-periodic by Claim 3.17), contradicting Claim 3.18. Therefore, $(K_{\bar{Y}'})^2 = 9$ or 8 and we may assume that $Y' = \mathbb{P}^2$ or $\mathbb{F}_d$, the Hirzebruch surface of degree $d \geq 0$.

If $Y' = \mathbb{P}^1 \times \mathbb{P}^1$, then $Y = Y'$, and G_0 stabilizes the fibers of the canonical projections $\pi_i : Y \to \mathbb{P}^1$ $(i = 1, 2)$ through a fixed point y_0 of $G_0|_Y$ (cf. Claim 3.19) and a section through y_0. But this contradicts Claim 3.18.

If $Y' = Y = \mathbb{F}_d$ for some $d \geq 1$, then G_0 stabilizes the fiber of the ruling $Y \to \mathbb{P}^1$ through a fixed point y_0 of $G_0|_Y$ and the unique $(-d)$-curve, contradicting Claim 3.18 again.

Therefore, either $Y = Y' = \mathbb{P}^2$ or $\mathbb{F}_d = Y' \to Y$ $(d \geq 1)$ is the contraction of the unique $(-d)$-curve. Thus $\rho(X_m) = 1 + \rho(Y) = 2$.

Let $D'_{ij} \subseteq X'$ be the proper transform of $D_{ij} \subseteq X_s$. Then G_0 also acts trivially on D'_{ij}. Note that every fiber of $\pi' : X' \to Y'$ is of dimension 1, the image $C_{ij} \subseteq Y'$ of D'_{ij} is the whole of Y' or a curve, and $G_0|_{C_{ij}} = \{\mathrm{id}\}$. By Claim 3.18, $G_0|_Y \neq \{\mathrm{id}\}$. So C_{ij} are curves in Y'. However, if $Y' = Y = \mathbb{P}^2$, then $G_0|_Y$ stabilizes C_{ij}; if $Y' = \mathbb{F}_d \to Y$ $(d \geq 1)$ is the contraction of the $(-d)$-curve, then $G_0|_Y$ stabilizes the images of C_{ij} on Y. Both contradicts Claim 3.18. Claim 3.21 is proved. □

We return to the proof of Theorem 3.6. Now can apply MMP to X_{s+1} and continue the process to reach an extremal Fano fibration $X_m \to Y$ so that the induced action of G on each X_i and on Y is regular, which is a contradiction (cf. Claim 3.17). We have completed the proof of Theorem 3.6. □

Corollary 3.22. *Let X be a projective manifold of dimension 3 and $G \leq \mathrm{Aut}(X)$ a subgroup of null entropy. Suppose that $G_0 := G \cap \mathrm{Aut}_0(X)$ is infinite and the quotient* *group G/G_0 is an almost abelian group of rank $r > 0$. Then (X, G) is not strongly primitive.*

Proof. Assume to the contrary that (X, G) is strongly primitive. Then by Theorem 3.6, X is a complex torus. Replacing G_0 by the identity component of its Zariski-closure in $\mathrm{Aut}_0(X)$ (and also replacing G), we may assume that G_0 is connected and dominating X. Then $G_0 = \mathrm{Aut}_0(X)$. It follows from Lemma 3.10 that $G/\mathrm{Aut}_0(G)$ is virtually abelian. Replacing G by a finite-index subgroup if necessary, we may assume that $G/G_0 = \langle \bar{g}_1 \rangle \oplus \cdots \oplus \langle \bar{g}_r \rangle$ for some $g_i \in G$. By Kronecker's theorem, as in the proof of [45, Lemma 2.14] $g_i^{s_i}$ has unipotent representation matrix on $H^0(X, \Omega_X^1)$. Again, replacing g_i by its power and G a finite-index subgroup, we may further assume that g_i has the unipotent representations.

Recall that $\mathrm{Aut}_{\mathrm{variety}}(X) = T \rtimes \mathrm{Aut}_{\mathrm{group}}(X)$. We can decompose $g_i = T_{t_i} \circ h_i$, where T_{t_i} is the translation by t_i and h_i is a group isomorphism. As in the proof of [45, Lemma 2.15], the h_i-fixed locus X^{h_i} is of dimension 1 or 2. Let B be the identity component of X^{h_1}. Note that $\bar{h}_1\bar{h}_j = \bar{h}_j\bar{h}_1$ in $G/\mathrm{Aut}_0(X)$. Since h_1h_j and h_jh_1 fix the origin, we have $h_1h_j = h_jh_1$ in G. It follows that $h_j(B) \subseteq X^{h_1}$, and hence $h_j(B) = B$ since both of them contain the origin. Therefore, $g_j(x + B) = g_j(x) + B$.

Clearly the elements of $\mathrm{Aut}_0(X)$ permutes the cosets of the quotient torus X/B; and we have shown that the same is true for g_j's. Therefore, $X \to X/B$ would be G-equivariant. This proves Corollary 3.22. ☐

Corollary 3.23. *Let X be a projective manifold of dimension 3 and $G \leq \mathrm{Aut}(X)$ a* *subgroup of null entropy such that* $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ *is almost abelian of rank* $r > 0$. *Assume that either* $\mathrm{Aut}_0(X) \neq 1$ *or the irregularity* $q(X) > 0$. *Then* (X, G) *is not strongly primitive.*

Proof. Assume that (X, G) is strongly primitive. If $q(X) > 0$, as in the proof of [44, Lemma 2.13], we may assume that X is a complex torus so that $\mathrm{Aut}_0(X) \neq 1$. So we may always assume that $\mathrm{Aut}_0(X) \neq 1$.

Replacing G by $G\ \mathrm{Aut}_0(X)$, we may assume that $G \geq G_0 := \mathrm{Aut}_0(X)$. Recall the exact sequence as in Lemma 3.8:

$$1 \to K \to G \to G|_{\mathrm{NS}_{\mathbb{C}}(X)} \to 1.$$

We have $G|_{\mathrm{NS}_{\mathbb{C}}(X)} \cong G/K$, with $|K : G_0| < \infty$. Then G/G_0 is almost abelian of rank $r > 0$ by our assumption. Corollary 3.23 thus follows from Corollary 3.22. $\square$

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